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Exterior products of zero-cycles

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Abstract. We study the exterior product $CH_0(X) \otimes CH_0(Y) \to CH_0(X \times Y)$ on 0-cycles modulo rational equivalence. The main tools used are higher cycle- and Abel-Jacobi-classes developed in [L1] and [K2]. The theorem of [RS] (applied to 0-cycles) appears as a special case of our results.

1. Introduction

Since Jannsen's formal definition [J4] of a conjectural *Bloch-Beilinson filtration* on the Chow groups of smooth projective varieties, a number of candidates have been put forth in the algebraic geometry literature. Those of Murre [M] and S. Saito [sS] are purely geometric, given in terms of the action of correspondences on cycles. Raskind's approach [Ra] is arithmetic, pulling back a filtration on continuous étale cohomology (arising from the Hochschild-Serre spectral sequence) along Jannsen's cycle-class map. On the other hand, Griffiths-Green [GG1], Lewis [L1] and M. Saito [mS] favor a Hodge-theoretic approach, using the Deligne-class map to pull back a Leray filtration on cohomology to the Chow group; central here is the idea of *spreading out* a cycle.

Under reasonable conjectural assumptions, these filtrations not only all yield Bloch-Beilinson filtrations—they all coincide (e.g., see [K1]). However, they are still quite useful in the absence of these assumptions, for instance in detecting (rational-equivalence classes of) cycles in the kernel of the Abel-Jacobi map. Exterior products of homologically trivial cycles, yield such cycles; and in this paper we turn our attention to the simplest case: products $\mathscr{Z}_1 \times \mathscr{Z}_2$ of degree-zero 0-cycles, considered on the product of the varieties on which they lie individually.

Torsion is excluded from our considerations: cycle (and thus Chow) groups are taken with rational coefficients; while Jacobians (such as the Albanese) are always of the rational sort, e.g. $\mathrm{Alb}(X) = \dfrac{H^{1,0}(X,\mathbb{C})^\vee}{\mathrm{im}\{H_1(X,\mathbb{Q})\}}$ for X smooth projective. (See §2.1 for a discussion of Jacobians.) In general a nonzero cycle, class, or invariant is always nontorsion. Rational equivalence is denoted by $\overset{\mathrm{rat}}{\equiv}$, and the $\overset{\mathrm{rat}}{\equiv}$-class of a cycle $\mathscr{Z} \in Z^p(X)$ is denoted by $\langle \mathscr{Z} \rangle \in CH^p(X)$ (to distinguish from its fundamental class $[\mathscr{Z}] \in \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-p), H^{2p}(X,\mathbb{Q})\big)$).

We now give a brief overview of the paper. Let Y_1, Y_2 be smooth projective varieties of respective dimensions $(d_1 =)\ 1$, d_2; and let $\langle \mathscr{Z}_2 \rangle \in CH_0^{\text{hom}}(Y_2)$ be such that the Abel-Jacobi image $AJ(\mathscr{Z}_2)$ is nontorsion in the Jacobian $J^{d_2}(Y_2) (= \text{Alb}(Y_2))$. By the result of Rosenschon-Saito [RS] one knows that if Y_1, Y_2, $\mathscr{Z}_2$ are all defined over $K \subseteqq \mathbb{C}$ (say, finitely generated $/\overline{\mathbb{Q}}$) but $\langle \mathscr{Z}_1 \rangle \in CH_0^{\text{hom}}((Y_1)_{\mathbb{C}})$ is not defined $/\bar{K}$ (i.e., $\langle \mathscr{Z}_1 \rangle \notin \text{im}\{CH_0((Y_1)_{\bar{K}}) \hookrightarrow CH_0((Y_1)_{\mathbb{C}})\}$), then $\langle \mathscr{Z}_1 \times \mathscr{Z}_2 \rangle$ is nonzero in $CH_0(Y_1 \times Y_2)$. The prototypical example is the 0-cycle $(P_1 - O_1) \times (P_2 - O_2)$ on a product of curves $\mathscr{C}_1$, $\mathscr{C}_2$ defined $/\overline{\mathbb{Q}}$, where $P_1 \in \mathscr{C}_1(\mathbb{C})$ is very general while $O_1 \in \mathscr{C}_1(\overline{\mathbb{Q}})$, $P_2, O_2 \in \mathscr{C}_2(\overline{\mathbb{Q}})$ and $AJ(P_2 - O_2)$ is nontorsion.

Our Theorem 1 generalizes their result to $d_1 > 1$, replacing the above condition on $\mathscr{Z}_1$ by (essentially) the requirement that its $\bar{K}$-spread $\overline{\mathfrak{Z}_1} \in Z^{d_1}(Y_1 \times \mathscr{S}_1)$ induce a nontrivial map of holomorphic forms $\Omega^j(Y_1) \to \Omega^j(\mathscr{S}_1)$ for some $1 \leqq j \leqq d_1$. Indeed, if $d_1 = 1$ and $\overline{\mathfrak{Z}_1}$ induces the zero map $\Omega^1(Y_1) \to \Omega^1(\mathscr{S}_1)$, then one can show that there exists a $\bar{K}$-specialization of $\mathscr{Z}_1$ having the same AJ class, hence (as $d_1 = 1 \Rightarrow Y_1$ is a curve) the same $\overset{\text{rat}}{\equiv}$-class—contradicting the [RS] condition on $\mathscr{Z}_1$. So for $d_1 = 1$ the conditions are equivalent. The result leads to a generalization of the "prototypical example" above to products of several curves; this was our original aim.

Returning to the Bloch-Beilinson picture, the filtration of Lewis leads to the higher cycle- and Abel-Jacobi-classes $cl^i(\cdot)$ and $AJ^i(\cdot)$ of [K2]; we describe how to compute these below. The point is that if we interpret Theorem 1 in terms of these invariants, it says (modulo GHC) that $cl^j_{Y_1}(\mathscr{Z}_1) \neq 0$ and $AJ^{(0)}_{Y_2}(\mathscr{Z}_2)[= \text{Alb}_{Y_2}(\mathscr{Z}_2)] \neq 0 \Rightarrow AJ^j_{Y_1 \times Y_2}(\mathscr{Z}_1 \times \mathscr{Z}_2) \neq 0$, where the latter AJ^j is computed by a sort of cup product of $cl^j_{Y_1}$ and AJ_{Y_2}. This points the way to the much broader generalization addressed in Theorem 2, where we start with nontrivial *higher* invariants $cl^{j_1}_{Y_1}(\mathscr{Z}_1)$ and $AJ^{j_2}_{Y_2}(\mathscr{Z}_2)$ for both cycles and ask when the "cup product" $AJ^{j_1+j_2}_{Y_1 \times Y_2}(\mathscr{Z}_1 \times \mathscr{Z}_2)$ (corresponding to the exterior product of cycles) is nontrivial. The answer involves a delicate quotient of the higher Abel-Jacobi class $AJ^{j_2}_{Y_2}(\mathscr{Z}_2)$ and careful consideration of the fields of definition of $\mathscr{Z}_1$ and $\mathscr{Z}_2$. (In fact, there are two different "quotients" obtained by successive projections $AJ^j(\mathscr{Z}) \mapsto AJ^j(\mathscr{Z})^{\text{tr}} \mapsto AJ^j(\mathscr{Z})^{\text{sf}}$; see eqn. (4).)

Here is a more precise statement of our main results: let Y_1, Y_2 be smooth projective (of any dimensions d_1, d_2) and defined $/\overline{\mathbb{Q}}$, and $\mathscr{L}^\bullet$ denote Lewis's filtration on CH_0.

Theorem 1′. *Given*

(a) *a field $K \subseteqq \mathbb{C}$ finitely generated $/\overline{\mathbb{Q}}$ (set $j := \text{trdeg}(K/\overline{\mathbb{Q}})$),*

(b) *$\langle \mathscr{Z}_1 \rangle \in \mathscr{L}^j CH_0((Y_1)_K)$ with complete $\overline{\mathbb{Q}}$-spread $\overline{\mathfrak{Z}_1} \in Z^{d_1}((Y_1 \times \mathscr{S})_{\overline{\mathbb{Q}}})$ inducing a nonzero map $\Omega^j(Y_1) \to \Omega^j(\mathscr{S})$,*

(c) *$\langle \mathscr{Z}_2 \rangle \in CH_0^{\text{hom}}((Y_2)_{\overline{\mathbb{Q}}})$ with nontorsion Albanese class in* $\text{Alb}(Y_2)$.

Then $\mathscr{Z} := \mathscr{Z}_1 \times \mathscr{Z}_2 \overset{\text{rat}}{\not\equiv} 0$ in $\mathscr{L}^{j+1} CH_0((Y_1 \times Y_2)_K)$; in particular, $AJ^j_{Y_1 \times Y_2}(\mathscr{Z})^{\text{tr}} \neq 0$.

The two corollaries provide various extensions—to the case where $j < \text{trdeg}(K/\overline{\mathbb{Q}})$, or where a second finitely generated field L takes the place of $\overline{\mathbb{Q}}$.

Here is a simple consequence of this theorem (from Example 2 below). If $\mathscr{C}_i/\bar{\mathbb{Q}}$ $(i = 1, \ldots, m+1)$ are smooth projective curves with $p_i \in \mathscr{C}_i(\mathbb{C})$ very general points, $o_i \in \mathscr{C}_i(\bar{\mathbb{Q}})$ $(i = 1, \ldots, m)$, and $\mathscr{W} \in CH_0^{\mathrm{hom}}((\mathscr{C}_{i+1})_{\bar{\mathbb{Q}}})$ has nontorsion AJ-class in $J^1(\mathscr{C}_{m+1})$, then $(p_1 - o_1) \times \cdots \times (p_m - o_m) \times \mathscr{W} \overset{\mathrm{rat}}{\not\equiv} 0$. This was previously known only for $m = 1$ (by [RS]).

If Theorem 1′ represents an application of new invariants to an outstanding problem, the next theorem can be seen as a statement about the behavior of the invariants themselves under exterior product.

Theorem 2′. *Given*

(a) $K_1, K_2 \subseteqq \mathbb{C}$ *f.g.* $/\bar{\mathbb{Q}}$ *with compositum* K *satisfying*

$$\mathrm{trdeg}(K_1/\bar{\mathbb{Q}}) + \mathrm{trdeg}(K_2/\bar{\mathbb{Q}}) = \mathrm{trdeg}(K/\bar{\mathbb{Q}}),$$

(b) $\langle \mathscr{Z}_1 \rangle \in \mathscr{L}^{j_1} CH_0((Y_1)_{K_1})$ *with* $cl_{Y_1}^{j_1}(\mathscr{Z}_1) \not\equiv 0$,

(c) $\langle \mathscr{Z}_2 \rangle \in \mathscr{L}^{j_2} CH_0((Y_2)_{K_2})$ *with either*

(i) $cl_{Y_2}^{j_2}(\mathscr{Z}_2) \not\equiv 0$ *or*

(ii) $AJ_{Y_2}^{j_2-1}(\mathscr{Z}_2)^{\mathrm{sf}} \not\equiv 0$ *and* $cl_{Y_2}^{j_2}(\mathscr{Z}_2) = \cdots = cl_{Y_2}^{d_2}(\mathscr{Z}_2) = 0$.

Assume the GHC.

Then $\mathscr{Z} := \mathscr{Z}_1 \times \mathscr{Z}_2 \overset{\mathrm{rat}}{\not\equiv} 0$ *in* $\mathscr{L}^{j_1+j_2} CH_0((Y_1 \times Y_2)_K)$. *In particular,*

when (i) *holds,* $cl_{Y_1 \times Y_2}^{j_1+j_2}(\mathscr{Z}) \not\equiv 0$;

when (ii) *holds,* $AJ_{Y_1 \times Y_2}^{j_1+j_2-1}(\mathscr{Z})^{\mathrm{tr}} \not\equiv 0$.

The proposition of §4 states what can be proved without the GHC, and has Theorem 1′ as the special case corresponding to (ii) with $j_2 = 1$.

In addition to these results there are several important lemmas which will be valuable in further applications (e.g. [K1], sec. 7).

Some additional notational remarks are in order: when a cycle (or variety) is denoted by a script (\mathcal) letter $\mathscr{Z}$, $\mathscr{V}$, $\mathscr{W}$, etc. (or a Roman letter), the corresponding gothic (\mathfrak) letter $\mathfrak{Z}$, $\mathfrak{V}$, $\mathfrak{W}$ always indicates its $\bar{\mathbb{Q}}$-spread; a bar over the latter denotes a choice of *complete* $\bar{\mathbb{Q}}$-spread (see §2.3). We have also replaced the notation $[AJ(\mathfrak{Z})]_i^{\mathrm{tr}}$ of [K1], sec. 6.1, by $[AJ(\mathfrak{Z})]_i^{\mathrm{sf}}$. In this paper, a line over a (complex) cohomology class (or subspace of a cohomology group) always denotes complex conjugation.

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2. Preliminaries

2.1. Hodge structures. A HS of weight m is a finite-dimensional $\mathbb{Q}$-vector-space $\mathscr{H}_{(\mathbb{Q})}$ with a filtration $F^{\bullet}$ on $\mathscr{H}_{\mathbb{C}} := \mathscr{H} \otimes_{\mathbb{Q}} \mathbb{C}$ such that

$$F^i \mathscr{H}_{\mathbb{C}} \oplus \overline{F^{m-i+1}\mathscr{H}_{\mathbb{C}}} = \mathscr{H}_{\mathbb{C}} = F^0 \mathscr{H}_{\mathbb{C}}.$$

We denote $F^i\mathscr{H}_{\mathbb{C}} \cap \overline{F^{m-i}\mathscr{H}_{\mathbb{C}}} =: \mathscr{H}^{i,m-i}_{(\mathbb{C})}$ and note $\mathscr{H}_{\mathbb{C}} = \bigoplus_{p+q=m} \mathscr{H}^{p,q}$. A $\mathbb{Q}$-subspace $\mathscr{G} \subseteqq \mathscr{H}$ is a subHS iff $\mathscr{G}_{\mathbb{C}} = \bigoplus_{p+q=m} \mathscr{G}^{p,q} := \bigoplus_{p+q=m} (\mathscr{G}_{\mathbb{C}} \cap \mathscr{H}^{p,q})$. Intersections and sums of subHS are subHS. Moreover, the quotient $\mathscr{H}/\mathscr{G}$ has a natural HS since $\mathscr{H}_{\mathbb{C}}/\mathscr{G}_{\mathbb{C}} \cong \bigoplus(\mathscr{H}^{p,q}/\mathscr{G}^{p,q})$. We write $\mathbb{Q}(-d)$ for the 1-dimensional weight $2d$ HS of pure type (d,d), and $F^i_h\mathscr{H}_{(\mathbb{Q})}$ for the largest subHS of $\mathscr{H}_{(\mathbb{Q})}$ contained in $F^i\mathscr{H}_{\mathbb{C}} \cap \mathscr{H}_{\mathbb{Q}}$ (equality is only true in general for $m = 2i$).

For $\mathscr{S}$ a smooth projective variety of dimension d over a field $K \subseteqq \mathbb{C}$, we write $H^m(\mathscr{S})$ for the HS $H^m_{\mathrm{sing}}(\mathscr{S}^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$. The fundamental class $[\mathscr{Z}]$ of an algebraic cycle $\mathscr{Z} \in Z^p(\mathscr{S})$ gives a subHS $\mathbb{Q}[\mathscr{Z}] \subseteqq H^{2p}(\mathscr{S})$. Note that for our purposes, $[\mathscr{Z}]$ may be defined by integration and Poincaré duality: $\int_{\mathscr{Z}} (\cdot) \mapsto [\mathscr{Z}]$ under the identification $\{H^{2d-2p}(\mathscr{S},\mathbb{C})\}^{\vee} \xrightarrow{\cong} H^{2p}(\mathscr{S},\mathbb{C}) \hookleftarrow \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-p), H^{2p}(\mathscr{S},\mathbb{Q})\big)$. (It generates a subHS because it is in fact a rational (p,p) class.) If $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$, the direct summands under the Künneth decomposition $H^m(\mathscr{S}) = \bigoplus_{r+s=m} H^r(\mathscr{S}_1) \otimes H^s(\mathscr{S}_2)$ are subHS; and the Künneth components $[\mathscr{Z}]_r$ of $[\mathscr{Z}]$ give subHS $\mathbb{Q}[\mathscr{Z}]_r \subseteqq H^r(\mathscr{S}_1) \otimes H^{2p-r}(\mathscr{S}_2)$. (Of course tensor products of HS have natural HS.)

For Y smooth quasiprojective, $H^i(Y)$ is in general a *mixed* Hodge structure. One can obtain a HS of weight i by setting $\underline{H^i}(Y) := \mathrm{im}\{H^i(\overline{Y}) \to H^i(Y)\} = W_iH^i(Y)$; this is independent of the choice of smooth compactification $\overline{Y}$.

The Generalized Hodge Conjecture $\mathrm{GHC}(i,m,\mathscr{S})$ predicts that

$$F^i_hH^m(\mathscr{S}) = N^iH^m(\mathscr{S}),$$

where $N^{\bullet}$ is the filtration by coniveau. The F^i_h do not behave so well under the Künneth decomposition: e.g.,

$$F^1_h\big(H^r(\mathscr{S}_1) \otimes H^s(\mathscr{S}_2)\big) \supseteqq F^1_hH^r(\mathscr{S}_1) \otimes H^s(\mathscr{S}_2) + H^r(\mathscr{S}_1) \otimes F^1_hH^s(\mathscr{S}_2)$$

may be a proper inclusion. We will also need the following "skew" subHS:

$$SF^{(i,j)}_h\big(H^r(\mathscr{S}_1) \otimes H^s(\mathscr{S}_2)\big) := \text{the largest subHS of } H^r(\mathscr{S}_1) \otimes H^s(\mathscr{S}_2)$$
$$\text{contained in } SF^{(i,j)}\big(H^r(\mathscr{S}_1,\mathbb{C}) \otimes H^s(\mathscr{S}_2,\mathbb{C})\big) \cap [H^r(\mathscr{S}_1) \otimes H^s(\mathscr{S}_2)]$$

where

$$SF^{(i,j)}\big(H^r(\mathscr{S}_1,\mathbb{C}) \otimes H^s(\mathscr{S}_2,\mathbb{C})\big)$$
$$:= F^iH^r(\mathscr{S}_1,\mathbb{C}) \otimes F^jH^s(\mathscr{S}_2,\mathbb{C}) + \overline{F^iH^r(\mathscr{S}_1,\mathbb{C})} \otimes \overline{F^jH^s(\mathscr{S}_2,\mathbb{C})}.$$

Note that $SF_h^{(1,\ell)}$ (for $\frac{s+1}{2} \geqq \ell \geqq 0$, $r > 0$) contains

$$N^1 H^r(\mathscr{S}_1) \otimes H^s(\mathscr{S}_2) + H^r(\mathscr{S}_1) \otimes F_h^\ell H^s(\mathscr{S}_2)$$

(since $N^1 \subseteqq F_h^1 \subseteqq F_{\mathbb{C}}^1 \cap \overline{F_{\mathbb{C}}^1}$, $F^\ell H_{\mathbb{C}}^s + \overline{F^\ell H_{\mathbb{C}}^s} = H_{\mathbb{C}}^s$, etc.).

A morphism of HS $\mathscr{H} \xrightarrow{\theta} {}'\mathscr{H}$ is a $\mathbb{Q}$-linear map which over $\mathbb{C}$ takes the form $\bigoplus \mathscr{H}^{p,q} \xrightarrow{\oplus \theta^{p,q}} \bigoplus {}'\mathscr{H}^{p,q}$ relative to a pair of bases subordinate to the resp. Hodge decompositions. Images and preimages of HS under such a morphism are HS.

We will use systematically the following notion of a "relative dual pair" of HS. This consists of:

(a) $\mathscr{K}_1 \subseteqq \mathscr{K}_0$ of weights $2d - 2n + 1$, $\mathscr{H}_1 \subseteqq \mathscr{H}_0$ of weights $2n - 1$, and

(b) a perfect pairing $\mathscr{H}_0 \times \mathscr{K}_0 \to \mathbb{Q}(-d)$ whose restriction to $\mathscr{H}_1 \times \mathscr{K}_1 \to \mathbb{Q}(-d)$ is also a perfect pairing.

The inclusions (in (a)) induce (by the duality in (b)) projections $\mathrm{pr}_{\mathscr{H}} : \mathscr{H}_0 \twoheadrightarrow \mathscr{H}_1$, $\mathrm{pr}_{\mathscr{K}} : \mathscr{K}_0 \twoheadrightarrow \mathscr{K}_1$, and the compositions $\mathscr{H}_1 \subseteqq \mathscr{H}_0 \twoheadrightarrow \mathscr{H}_1$, $\mathscr{K}_1 \subseteqq \mathscr{K}_0 \twoheadrightarrow \mathscr{K}_1$ yield the respective identity maps. (One should view, for example, $\mathscr{H}_1 \hookrightarrow \mathscr{H}_0$ as [a choice of] extension of functionals from $\mathscr{K}_1$ to $\mathscr{K}_0$, and $\mathscr{H}_0 \twoheadrightarrow \mathscr{H}_1$ as restriction of functionals from $\mathscr{K}_0$ to $\mathscr{K}_1$.) Equivalently, one could replace (b) with:

(b′) identifications $\mathscr{H}_i \cong \mathscr{K}_i^\vee \otimes \mathbb{Q}(-d)$ (which together with (a) induce $\mathrm{pr}_{\mathscr{H}}$, $\mathrm{pr}_{\mathscr{K}}$) such that the above compositions give the identity.

In this paper the pairings always come tacitly from Poincaré duality. Note that $\mathrm{pr}_{\mathscr{H}}$, $\mathrm{pr}_{\mathscr{K}}$ have kernels $\mathscr{H}_1'$, $\mathscr{K}_1'$ (resp.) which satisfy: $\mathscr{H}_0 = \mathscr{H}_1 \oplus \mathscr{H}_1'$, $\mathscr{K}_0 = \mathscr{K}_1 \oplus \mathscr{K}_1'$. (This approach to complementary HS's gives us more control than using the semisimplicity coming from a polarization.)

If $\mathscr{H}$ is of weight $2n - 1$, define the Jacobian $J^n(\mathscr{H}) := \mathscr{H}_{\mathbb{C}}/(F^n \mathscr{H}_{\mathbb{C}} + \mathscr{H}_{\mathbb{Q}})$; and if $\mathscr{H} = \mathscr{K}^\vee \otimes \mathbb{Q}(-d)$, then $J^n(\mathscr{H}) = (F^{d-n+1}\mathscr{K}_{\mathbb{C}})^\vee / \mathscr{K}_{\mathbb{Q}}^\vee$. When $\mathscr{H}_1 \subseteqq \mathscr{H}_0$ is a subHS, $J^n(\mathscr{H}_1) \hookrightarrow J^n(\mathscr{H}_0)$ and $J^n(\mathscr{H}_0) \twoheadrightarrow J^n(\mathscr{H}_0/\mathscr{H}_1) \cong J^n(\mathscr{H}_0)/J^n(\mathscr{H}_1)$. In the above "dual pair" situation, $J(\mathscr{H}_0) = J(\mathscr{H}_1) \oplus J({}'\mathscr{H}_1)$; we emphasize that since extension followed by restriction of functionals $\mathscr{H}_1 \subseteqq \mathscr{H}_0 \twoheadrightarrow \mathscr{H}_1$ is the identity, so is the induced composition on Jacobians. More generally a morphism θ induces a map of Jacobians. We write $J^p\big(H^{2p-1}(\mathscr{S})\big) =: J^p(\mathscr{S})$.

To construct elements in Jacobians: let $\dim(\mathscr{S}) = d$, $\mathscr{Z} \in Z^p_{\mathrm{hom}}(\mathscr{S})$, $\partial^{-1}\mathscr{Z} =$ any choice of topological $(2d - 2p + 1)$-chain bounding on $\mathscr{Z}$, and let $\mathscr{K} \subseteqq H^{2d-2p+1}(\mathscr{S})$, $\mathscr{H} \subseteqq H^{2p-1}(\mathscr{S})$ be a relative dual pair. Then $\int_{\partial^{-1}\mathscr{Z}} (\cdot)$ is a well-defined functional on $F^{d-p+1}H^{2d-2p+1}(\mathscr{S}, \mathbb{C})$. (Represent the latter by C^∞ forms $\omega \in F^{d-p+1}\Omega^{2d-2p+1}_{\mathscr{S}^\infty}(\mathscr{S})$; note that if $\omega - \omega' = d\alpha$, α may be chosen $\in F^{d-p+1}$, and use Stokes's theorem.) Hence one has an element of $\{F^{d-p+1}\mathscr{K}_{\mathbb{C}}\}^{\vee} \twoheadrightarrow J^p(\mathscr{H})$; we write $\left(\int\limits_{\partial^{-1}\mathscr{Z}} (\cdot)\right) \in J^p(\mathscr{H})$.

2.2. The fundamental lemma. This is the main organizational tool for the proofs of Theorem 1 (§3) and the proposition of §4. For clarity, we break it into the cases (i) and (ii) needed in the respective proofs of these main results, even though (i) is a special case of (ii) (put $\mathscr{G}_0 = \{0\}$).

Lemma 1. *Given the following* 3 *items*:

(a) $\mathscr{H}_1 \subsetneqq \mathscr{H}_0$, $\mathscr{K}_1 \subsetneqq \mathscr{K}_0$ *a relative dual pair of* HS, *with* $\mathscr{G}_0 \subsetneqq \mathscr{H}_0$ *a* subHS *mapped into itself under the composition* $\mathscr{H}_0 \twoheadrightarrow \mathscr{H}_1 \subsetneqq \mathscr{H}_0$;

(b) $\mathscr{H}_{\mathfrak{B}} \subsetneqq \mathscr{H}_1$, $\mathscr{K}_{\mathfrak{B}} \subsetneqq \mathscr{K}_1$ *a relative dual pair, with* $\mathscr{H}_2$ *and* $\mathscr{K}_2$ *denoting the respective projection-kernels*: $\mathscr{H}_1 = \mathscr{H}_{\mathfrak{B}} \oplus \mathscr{H}_2$, *etc.*; *and*

(c) $\Xi \in J(\mathscr{H}_0)$ *with lifting* $\tilde{\Xi} \in \{F^{d-n+1}\mathscr{K}_0^{\mathbb{C}}\}^{\vee}$, *and* $\xi \in J(\mathscr{H}_{\mathfrak{B}})$ *with lifting* $\tilde{\xi} \in \{F^{d-n+1}\mathscr{K}_{\mathfrak{B}}^{\mathbb{C}}\}^{\vee}$, *such that* $\tilde{\Xi}$ *is trivial on* $F^{d-n+1}\mathscr{K}_2^{\mathbb{C}}$ $(= \mathscr{K}_2^{\mathbb{C}} \cap F^{d-n+1}\mathscr{K}_0^{\mathbb{C}})$ *and equivalent to* $\tilde{\xi}$ *on* $F^{d-n+1}\mathscr{K}_{\mathfrak{B}}^{\mathbb{C}}$.

Now assume also that one of the following is satisfied:

(i) $\xi \neq 0$ *and* $\mathscr{H}_{\mathfrak{B}} \cap (\mathscr{H}_1 \cap \mathscr{G}_0) = \{0\}$; *or*

(ii) $\exists$subHS $\mathscr{G}_1 \subsetneqq \mathscr{H}_1$ *containing* $\mathscr{H}_{\mathfrak{B}} \cap (\mathscr{H}_1 \cap \mathscr{G}_0)$ *and such that the projection of* ξ *to* $J\big(\mathscr{H}_{\mathfrak{B}}/(\mathscr{G}_1 \cap \mathscr{H}_{\mathfrak{B}})\big)$ *is nonzero.*

Then the projection of Ξ *to* $J(\mathscr{H}_0/\mathscr{G}_0)$ *is nonzero.*

Proof. From (a) and (b) one has the diagram

$$\begin{tikzcd}
 & \dfrac{\{F^{d-n+1}\mathscr{K}_0^{\mathbb{C}}\}^{\vee}}{\{\mathscr{K}_0^{\mathbb{Q}}\}^{\vee}} \arrow[r, leftrightarrow, "\cong"] & J(\mathscr{H}_0) \arrow[r, two heads, "\beta_0"] \arrow[d, "\mathrm{pr}_1"] & J\left(\dfrac{\mathscr{H}_0}{\mathscr{G}_0}\right) \arrow[d, "\overline{\mathrm{pr}_1}"] \\
\dfrac{\{F^{d-n+1}\mathscr{K}_{\mathfrak{B}}^{\mathbb{C}}\}^{\vee}}{\{\mathscr{K}_{\mathfrak{B}}^{\mathbb{Q}}\}^{\vee}} \arrow[r, leftrightarrow, "\cong"] & J(\mathscr{H}_{\mathfrak{B}}) \arrow[r, hook, "\imath_{\mathfrak{B}}"] & J(\mathscr{H}_1) \arrow[r, two heads, "\beta_1"] \arrow[d, "\cong"', "(\mathrm{pr}_{\mathfrak{B}}{,}\,\mathrm{pr}_2)"] & J\left(\dfrac{\mathscr{H}_1}{\mathscr{H}_1 \cap \mathscr{G}_0}\right) \arrow[d, equal] \\
 & & J(\mathscr{H}_{\mathfrak{B}}) \oplus J(\mathscr{H}_2) \arrow[ul] & \dfrac{J(\mathscr{H}_1)}{J(\mathscr{H}_1 \cap \mathscr{G}_0)}
\end{tikzcd}$$

in which the square commutes and $\mathrm{pr}_{\mathfrak{B}} \circ \imath_{\mathfrak{B}}$ is the identity. From (c), $(\mathrm{pr}_{\mathfrak{B}} \circ \mathrm{pr}_1)(\Xi) = \xi$ and $(\mathrm{pr}_2 \circ \mathrm{pr}_1)(\Xi) = 0$; hence $\mathrm{pr}_1(\Xi) = \imath_{\mathfrak{B}}(\xi)$.

If (i) holds, then $J(\mathscr{H}_{\mathfrak{B}}) \oplus J(\mathscr{H}_1 \cap \mathscr{G}_0) \hookrightarrow J(\mathscr{H}_1)$ and so β_1 cannot kill $\imath_{\mathfrak{B}}(\xi)$; we conclude that $\beta_0(\Xi) \neq 0$.

If (ii) holds, then $\pi_{\mathfrak{B}}(\xi) \neq 0$ in the diagram

$$\begin{array}{ccccc}
J(\mathscr{H}_{\mathfrak{B}}) & \xrightarrow{\iota_{\mathfrak{B}}} & J(\mathscr{H}_1) & \xrightarrow{\beta_1} & J\big(\mathscr{H}_1/(\mathscr{H}_1 \cap \mathscr{G}_0)\big) \\
\downarrow{\scriptstyle \pi_{\mathfrak{B}}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{\pi}} \\
J\big(\mathscr{H}_{\mathfrak{B}}/(\mathscr{G}_1 \cap \mathscr{H}_{\mathfrak{B}})\big) & \xrightarrow{\overline{\iota_{\mathfrak{B}}}} & J(\mathscr{H}_1/\mathscr{G}_1) & \xrightarrow{\overline{\beta_1}} & J\big(\mathscr{H}_1/\{(\mathscr{H}_1 \cap \mathscr{G}_0) + \mathscr{G}_1\}\big) \\
 & & & & \| \\
 & & & & \dfrac{J(\mathscr{H}_1/\mathscr{G}_1)}{J\big((\mathscr{H}_1 \cap \mathscr{G}_0)/(\mathscr{G}_1 \cap \mathscr{G}_0)\big)}
\end{array}$$

(in which squares commute). Moreover, $\mathscr{H}_{\mathfrak{B}} \cap (\mathscr{H}_1 \cap \mathscr{G}_0) \subseteq \mathscr{G}_1$ implies that

$$\frac{\mathscr{H}_1 \cap \mathscr{G}_0}{\mathscr{G}_1 \cap \mathscr{G}_0} \cap \frac{\mathscr{H}_{\mathfrak{B}}}{\mathscr{G}_1 \cap \mathscr{H}_{\mathfrak{B}}} = \{0\} \quad \text{in } \frac{\mathscr{H}_1}{\mathscr{G}_1};$$

hence $J\left(\dfrac{\mathscr{H}_1 \cap \mathscr{G}_0}{\mathscr{G}_1 \cap \mathscr{G}_0}\right) \oplus J\left(\dfrac{\mathscr{H}_{\mathfrak{B}}}{\mathscr{G}_1 \cap \mathscr{H}_{\mathfrak{B}}}\right) \hookrightarrow J\left(\dfrac{\mathscr{H}_1}{\mathscr{G}_1}\right)$. Therefore $\overline{\beta_1}$ cannot kill $\overline{\iota_{\mathfrak{B}}}\big(\pi_{\mathfrak{B}}(\xi)\big)$, and so (by both diagrams) $\bar{\pi}\big(\overline{\mathrm{pr}_1}\big(\beta_0(\Xi)\big)\big) = \overline{\beta_1}\big(\overline{\iota_{\mathfrak{B}}}\big(\pi_{\mathfrak{B}}(\xi)\big)\big) \neq 0 \Rightarrow \beta_0(\Xi) \neq 0$ once again. $\square$

2.3. Points and spreads. Given $\mathscr{S}/\overline{\mathbb{Q}}$ smooth projective, choose any affine Zariski open subset $S/\overline{\mathbb{Q}}$. The embedding $\overline{\mathbb{Q}}[S] \hookrightarrow \overline{\mathbb{Q}}(S) \cong \overline{\mathbb{Q}}(\mathscr{S})$ then produces a generic point p_g on $\mathscr{S}$ via the composition $\operatorname{Spec} \overline{\mathbb{Q}}(\mathscr{S}) \to \operatorname{Spec} \overline{\mathbb{Q}}[S] \cong S \hookrightarrow \mathscr{S}$. For purposes of taking cohomology, we use the approximation $\eta_{\mathscr{S}} = \varprojlim \mathscr{U}$ (over $\mathscr{U} \subseteq \mathscr{S}$ affine Zariski open subsets defined $/\overline{\mathbb{Q}}$) to p_g; more precisely, $H^i(\eta_{\mathscr{S}}) := \varinjlim H^i(\mathscr{U}^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$ while

$$CH^p(X \times_{\overline{\mathbb{Q}}} \eta_{\mathscr{S}}) := \varinjlim CH^p(X \times_{\overline{\mathbb{Q}}} \mathscr{U}) \cong CH^p(X \times_{\overline{\mathbb{Q}}} p_g).$$

Note that $\underline{H}^i(\eta_{\mathscr{S}}) := \operatorname{im}\{H^i(\mathscr{S}) \to H^i(\eta_{\mathscr{S}})\} \cong H^i(\mathscr{S})/N^1 H^i(\mathscr{S})$ is a HS. (See [K2], sec. 3, 4, for important well-definedness checks.)

Given any embedding $\mathrm{ev} : \overline{\mathbb{Q}}(\mathscr{S}) \hookrightarrow \mathbb{C}$ which restricts to the identity on $\overline{\mathbb{Q}}$, define a geometric point $p \in \mathscr{S}(\mathbb{C})$ of maximal transcendence degree ($= \dim(\mathscr{S})$) over $\overline{\mathbb{Q}}$ by $p := p_g \times_{\mathrm{ev}} \operatorname{Spec} \mathbb{C}$. (See [K1]. In fact p is defined over $K := \mathrm{ev}\big(\overline{\mathbb{Q}}(\mathscr{S})\big)$.) Since p lies in the complement of the countably many divisors $D \subseteq \mathscr{S}$ defined $/\overline{\mathbb{Q}}$, we may think of it as a geometric (closed, zero-dimensional) point of $\eta^{\mathbb{C}}_{\mathscr{S}}$; such a point is called *very general.*

The *compositum* of two fields $K_1, K_2 \subseteq \mathbb{C}$ is the smallest subfield of $\mathbb{C}$ containing both K_1 and K_2; if $K_1, K_2 \supseteq \overline{\mathbb{Q}}$ then this may be written $\overline{\mathbb{Q}}(K_1, K_2)$. To define the $\overline{\mathbb{Q}}$-spread of an algebraic cycle we need part (a) of the following:

Lemma 2. (a) *Let $K \subseteq \mathbb{C}$ be a finitely generated extension of $\overline{\mathbb{Q}}$. Then $\exists \mathscr{S}/\overline{\mathbb{Q}}$ smooth projective and a very general point $p \in \mathscr{S}(\mathbb{C})$ such that $\mathrm{ev}_p : \overline{\mathbb{Q}}(\mathscr{S}) \xrightarrow{\cong} K$.*

(b) *Given $\overline{\mathbb{Q}}(\mathscr{S}_1) \cong K_1$, $\overline{\mathbb{Q}}(\mathscr{S}_2) \cong K_2$ two such,*

$$\overline{\mathbb{Q}}(\mathscr{S}_1 \times \mathscr{S}_2) \cong \overline{\mathbb{Q}}(K_1, K_2) \iff \operatorname{trdeg}\big(\overline{\mathbb{Q}}(K_1, K_2)/\overline{\mathbb{Q}}\big) = \operatorname{trdeg}(K_1/\overline{\mathbb{Q}}) + \operatorname{trdeg}(K_2/\overline{\mathbb{Q}}).$$

Proof. (a) $K = \overline{\mathbb{Q}}(\pi_1, \ldots, \pi_t; \alpha_1, \ldots, \alpha_s)$ for $\{\pi_i\}$ a transcendence basis $\Rightarrow$ we can map $\overline{\mathbb{Q}}[x_1, \ldots, x_t; x_{t+1}, \ldots, x_{t+s}] \xrightarrow{\phi} K$ via $\{x_i \mapsto \pi_i, x_{t+j} \mapsto \alpha_j\}$. Set $R = \mathrm{im}(\phi)$, $I = \ker(\phi)$, $S = \mathrm{Var}(I) \subseteqq \mathbb{A}^{t+s}_{\overline{\mathbb{Q}}}$. Since R is a division ring, I is prime and S irreducible, $\dim S = t = \mathrm{trdeg}(K/\overline{\mathbb{Q}})$. So $\bar{\phi} : \overline{\mathbb{Q}}[S] = \dfrac{\overline{\mathbb{Q}}[x_1, \ldots, x_{t+s}]}{I} \xrightarrow{\cong} R$ induces an isomorphism of fraction fields $\overline{\mathbb{Q}}(S) = \dfrac{\overline{\mathbb{Q}}(x_1, \ldots, x_t)[x_{t+1}, \ldots, x_{t+s}]}{I} \xrightarrow{\cong} K$; this is evaluation at $p = (\pi_1, \ldots, \pi_t; \alpha_1, \ldots, \alpha_s) \in S(\mathbb{C})$. Finally, take $\tilde{S}$ to be a desingularization of S, and $\mathscr{S}$ to be a good compactification of $\tilde{S}$; one has $\overline{\mathbb{Q}}(\mathscr{S}) \cong \overline{\mathbb{Q}}(\tilde{S}) \cong \overline{\mathbb{Q}}(S)$.

(b) As in (a), we have $p = p_1 \times p_2 = (\{\underline{\pi}\}, \{\underline{\alpha}\}; \{\underline{\sigma}\}, \{\underline{\beta}\}) \in S_1 \times S_2 \subseteqq \mathbb{A}^{t_1+s_1}_{\{x\}} \times \mathbb{A}^{t_2+s_2}_{\{y\}}$ for $\{\underline{\pi}\}$, $\{\underline{\sigma}\}$ transcendence bases for K_1, $K_2/\overline{\mathbb{Q}}$. Evaluation at p gives a map

$$\overline{\mathbb{Q}}[S_1 \times S_2] = \overline{\mathbb{Q}}[S_1] \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}[S_2] \cong \frac{\overline{\mathbb{Q}}[x_1, \ldots, x_{t_1+s_1}; y_1, \ldots, y_{t_2+s_2}]}{I = (I_1^{\{x\}}, I_2^{\{y\}})} \xrightarrow{\bar{\phi}} \overline{\mathbb{Q}}(K_1, K_2) \subseteqq \mathbb{C}.$$

If $\overline{\mathbb{Q}}(\mathscr{S}_1 \times \mathscr{S}_2)[\cong \overline{\mathbb{Q}}(S_1 \times S_2)]$ is not $\cong$ to the fraction field of $\mathrm{im}(\bar{\phi})$, then $\bar{\phi}$ kills some $f \not\equiv 0 \ (\mathrm{mod}\, I)$. Since I is prime, Nullstellensatz $\Rightarrow \mathscr{I}(S_1 \times S_2) = I$, hence f does not vanish on $S_1 \times S_2$ and $f = 0$ cuts out a subvariety $D/\overline{\mathbb{Q}}$ of codim. $\geqq 1$ in which p must sit. Since the relative dimension of $\mathrm{pr} : S_1 \times S_2 \to \mathbb{A}^{t_1}_{\{x\}} \times \mathbb{A}^{t_2}_{\{y\}}$ is 0, $\mathrm{pr}(p) = (\{\underline{\pi}\}, \{\underline{\sigma}\})$ sits in a $\overline{\mathbb{Q}}$-subvariety of $\mathbb{A}^{t_1+t_2}$ of codim. $\geqq 1$; therefore $\pi_1, \ldots, \pi_{t_1}; \sigma_1, \ldots, \sigma_{t_2}$ are not algebraically independent. But $\{\alpha_i, \beta_j\}$ are algebraic over $\overline{\mathbb{Q}}(\{\pi\}, \{\sigma\})$, hence $\overline{\mathbb{Q}}(K_1, K_2)$ has $\mathrm{trdeg} \leqq t_1 + t_2 - 1$. $\square$

Now let X be defined over $\overline{\mathbb{Q}}$, K be f.g. $/\overline{\mathbb{Q}}$, $X_K = X \otimes_{\overline{\mathbb{Q}}} \mathrm{Spec}\, K$ and $\mathscr{Z} \in Z^p(X_K)$. By Lemma 2(a) one has $\mathrm{ev}_p : \overline{\mathbb{Q}}(\mathscr{S}) \xrightarrow{\cong} K$, and we define

$$\mathscr{Z}_g := \mathscr{Z} \times_{\mathrm{ev}_p^{-1}} \mathrm{Spec}\, \overline{\mathbb{Q}}(\mathscr{S}) \in Z^p(X_{\overline{\mathbb{Q}}(\mathscr{S})}).$$

Clearing denominators from the equations cutting out the components of $\mathscr{Z}_g$ yields a cycle $\bar{\mathfrak{Z}} \in Z^p((X \times \mathscr{S})_{\overline{\mathbb{Q}}})$ whose complexification restricts to $\mathscr{Z}$ along $X \times \{p\} \hookrightarrow X \times \mathscr{S}$. (Such a "complete spread" is not well-defined modulo $\overset{\mathrm{rat}}{\equiv}$.) One also has the obvious restriction $\mathfrak{Z} \in Z^p((X \times \eta_{\mathscr{S}})_{\overline{\mathbb{Q}}})$ which is called the $\overline{\mathbb{Q}}$-spread of $\mathscr{Z}$. We can write this as a map

$$CH^p(X_K) \xrightarrow{\cong} CH^p((X \times \eta_{\mathscr{S}})_{\overline{\mathbb{Q}}}). \tag{1}$$

2.4. Higher cycle- and Abel-Jacobi classes. In order to define our invariants we need Deligne cohomology and the Deligne cycle-class (roughly an amalgamation of fundamental class and Abel-Jacobi class), for which the reader may consult [K1], §§2.4, 3.1. An expanded treatment of the Lewis filtration may be found in [K1], sec. 4.2, [K2], and of course [L1]. In this subsection (up to Lemma 3) we merely review those points which are required in the remainder of the paper.

To take the Deligne class of the r.h.s. of (1) we use the well-defined "composition" ψ:

$$CH^p((X \times \eta_{\mathscr{S}})_{\overline{\mathbb{Q}}}) \twoheadleftarrow CH^p((X \times \mathscr{S})_{\overline{\mathbb{Q}}}) \xrightarrow{c_{\mathscr{D}}} H^{2p}_{\mathscr{D}}(X \times \mathscr{S}, \mathbb{Q}(p)) \twoheadrightarrow \underline{H}^{2p}_{\mathscr{D}}(X \times \eta_{\mathscr{S}}, \mathbb{Q}(p))$$

(where $\underline{H_{\mathscr{D}}^{2p}}$ denotes the image of Deligne cohomology of $(X \times \mathscr{S})_{\mathbb{C}}^{\mathrm{an}}$ in absolute Hodge cohomology of $(X \times \eta_{\mathscr{S}})_{\mathbb{C}}^{\mathrm{an}}$, see [L1]). Write $\Psi^{K/\bar{\mathbb{Q}}} := \psi \circ (1)$.

Lewis [L1] constructs a (decreasing) Leray filtration $\mathscr{L}^{\bullet}$ on $\underline{H_{\mathscr{D}}^{2p}}(X \times \eta_{\mathscr{S}}, \mathbb{Q}(p))$. This uses a presentation (known as "Deligne homology") of $H_{\mathscr{D}}^{2p}(X \times \overline{\mathscr{S}}, \mathbb{Q}(p))$ as cohomology of a complex of triples of global chains and currents, which can then be filtered by degree along $\mathscr{S}$. Actually Lewis works in a more general situation; an explicit description of $\mathscr{L}^{\bullet}$ for our purposes here may be found in [K2], §10 (see eqn. (10.6)ff.). With this $\mathscr{L}^{\bullet}$ granted, taking ψ-preimages then automatically gives a decreasing filtration $\mathscr{L}^{\bullet}$ on both groups of (1).

In the notation of [K2], one has exact sequences

$$(2) \qquad 0 \to Gr_{\mathscr{L}}^{i-1} \underline{J^p}(X \times \eta_{\mathscr{S}}) \to Gr_{\mathscr{L}}^{i} \underline{H_{\mathscr{D}}^{2p}}(X \times \eta_{\mathscr{S}}, \mathbb{Q}(p)) \to Gr_{\mathscr{L}}^{i} \underline{Hg^p}(X \times \eta_{\mathscr{S}}) \to 0$$

with the identifications in (3) and (4) below. (This is proved in [L1] and also follows from eqn. (10.4) and its proof in [K2].) If $\langle \mathfrak{Z} \rangle \in \mathscr{L}^i CH^p((X \times \eta_{\mathscr{S}})_{\bar{\mathbb{Q}}})$ $[\Leftrightarrow \langle \mathscr{Z} \rangle \in \mathscr{L}^i CH^p(X_K)]$ then its invariants in (2) are written $[c_{\mathscr{D}}(\mathfrak{Z})]_i$ (or $\Psi_i^{K/\bar{\mathbb{Q}}}(\mathscr{Z})$) in the middle term, $[\mathfrak{Z}]_i$ in the r.h. term, and (if $[\mathfrak{Z}]_i = 0$) $[AJ(\mathfrak{Z})]_{i-1}$ on the left. One has

$$(3) \qquad Gr_{\mathscr{L}}^{i} \underline{Hg^p}(X \times \eta_{\mathscr{S}}) := \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(-p), \underline{H^i}(\eta_{\mathscr{S}}) \otimes H^{2p-i}(X)) \hookrightarrow \underline{H^i}(\eta_{\mathscr{S}}) \otimes H^{2p-i}(X),$$

$$(4) \qquad Gr_{\mathscr{L}}^{i-1} \underline{J^p}(X \times \eta_{\mathscr{S}}) := \frac{\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-p), \underline{H^{i-1}}(\eta_{\mathscr{S}}) \otimes H^{2p-i}(X))}{\mathrm{im}\{\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(-p), Gr_i^W H^{i-1}(\eta_{\mathscr{S}}) \otimes H^{2p-i}(X))\}}$$
$$\twoheadrightarrow J^p\left(\underline{H^{i-1}}(\eta_{\mathscr{S}}) \otimes \frac{H^{2p-i}(X)}{F_h^{p-i+1} H^{2p-i}(X)}\right)$$
$$\twoheadrightarrow J^p\left(\frac{H^{i-1}(\mathscr{S}) \otimes H^{2p-i}(X)}{SF_h^{(1,p-i+1)}\{\mathrm{num}\}}\right),$$

where "num" means "numerator". In (4), the second projection is valid by the remarks on SF in §2.1 for $i \geqq 2$, and trivially for $i = 1$; while the first projection is worked out in [K2], sec. 12. (Note that the $\mathrm{Ext}^1_{\mathrm{MHS}} \cong J^p(\underline{H^{i-1}}(\eta_{\mathscr{S}}) \otimes H^{2p-i}(X))$.) The successive projected images of $[AJ(\mathfrak{Z})]_{i-1}$ are written $[AJ(\mathfrak{Z})]_{i-1}^{\mathrm{tr}}$ and $[AJ(\mathfrak{Z})]_{i-1}^{\mathrm{sf}}$.

Let $\bar{\mathfrak{Z}}$ be a (choice of) complete spread. To compute $[\mathfrak{Z}]_i$ one takes the image of the Künneth component $[\bar{\mathfrak{Z}}]_i \in H^i(\mathscr{S}) \otimes H^{2p-i}(X)$ in the r.h. term of (3). If the full fundamental class $[\bar{\mathfrak{Z}}] = 0$ in $H^{2p}(X \times \mathscr{S})$, then one may compute $[AJ(\mathfrak{Z})]_{i-1}$ and its images by projecting $\left(\int_{\partial^{-1}\bar{\mathfrak{Z}}} (\cdot)\right) \in J^p(H^{i-1}(\mathscr{S}) \otimes H^{2p-i}(X))$ to the appropriate term in (4).

The original filtration of Lewis [L1] is (for X defined over $\bar{\mathbb{Q}}$ as considered here) obtained by taking a limit $\mathscr{L}^i CH^p(X_{\mathbb{C}}) := \varinjlim_K \mathscr{L}^i CH^p(X_K)$ over all $K \subseteqq \mathbb{C}$ finitely generated $/\bar{\mathbb{Q}}$. Note that $CH^p(X_K) \hookrightarrow CH^p(X_{\mathbb{C}})$, so that $\varinjlim_K CH^p(X_K) = CH^p(X_{\mathbb{C}})$. On the cohomology side, this corresponds to a limit over all finite-dimensional smooth projective varieties $\mathcal{S}/\bar{\mathbb{Q}}$ and dominant morphisms $\mathcal{S}' \to \mathcal{S}$ (these induce injective maps of $Gr^i_{\mathcal{L}}\underline{H^{2p}_{\mathcal{D}}}\big(X \times \eta_{(\cdot)}, \mathbb{Q}(p)\big)$, see [K2], sec. 3). One has maps

$$\Psi_i^{(\mathbb{C}/\bar{\mathbb{Q}})} : \mathcal{L}^i CH^p(X_{\mathbb{C}}) \to \varinjlim_{\mathcal{S}} Gr^i_{\mathcal{L}}\underline{H^{2p}_{\mathcal{D}}}\big(X \times \eta_{\mathcal{S}}, \mathbb{Q}(p)\big)$$

and exact sequences $\varinjlim_{\mathcal{S}}(2)$ obtained simply by applying the direct limit (exact in the category of Abelian groups) to each term of (2). Given $\langle \mathcal{Z} \rangle \in \mathcal{L}^i$, one writes invariants $cl^i_X(\mathcal{Z}) \in \varinjlim_{\mathcal{S}} Gr^i_{\mathcal{L}}\underline{Hg^p}(X \times \eta_{\mathcal{S}})$ and (if this $= 0$) $AJ^{i-1}_X(\mathcal{Z}) \in \varinjlim_{\mathcal{S}} Gr^{i-1}_{\mathcal{L}}\underline{J^p}(X \times \eta_{\mathcal{S}})$ (which are essentially $[\mathfrak{Z}]_i$ and $[AJ(\mathfrak{Z})]_{i-1}$ but without referring to K or $\mathfrak{Z}$).

We will use the following two lemmas in the proofs of Corollary 1 and Theorem 2. In the first one (writing $d = \dim X$) we take $p = d$, which corresponds to the case where $\mathcal{Z}$ is a 0-cycle.

Lemma 3. *Let $\mathfrak{Z} \in \mathcal{L}^i CH^d\big((X \times \eta_{\mathcal{S}})_{\bar{\mathbb{Q}}}\big)$. If $[\mathfrak{Z}]_i \neq 0$ and* GHC$(1, i, \mathcal{S})$ *holds then the induced map $\bar{\mathfrak{Z}}^* : \Omega^i(X) \to \Omega^i(\mathcal{S})$ is nontrivial.*

Proof. Noting that $N^1H^i(\mathcal{S}) \subseteq F^1_h H^i(\mathcal{S}) \Rightarrow \underline{H^{i,0}}(\eta_{\mathcal{S}}, \mathbb{C}) = H^{i,0}(\mathcal{S}, \mathbb{C})$, we must show the composition ϕ:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-d), \underline{H^i}(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\big) &\hookrightarrow \underline{H^i}(\eta_{\mathcal{S}}, \mathbb{C}) \otimes H^{2d-i}(X, \mathbb{C}) \\ &\twoheadrightarrow H^{i,0}(\mathcal{S}, \mathbb{C}) \otimes H^{d-i,d}(X, \mathbb{C})\end{aligned}$$

is injective. An element of the $\mathrm{Hom}_{\mathrm{MHS}}$ is a morphism $\theta : H^i(X) \to \underline{H^i}(\eta_{\mathcal{S}})$ of HS; if $\theta \in \ker(\phi)$ then

$$\mathrm{im}(\theta) \subseteq \ker\{\underline{H^i}(\eta_{\mathcal{S}}) \hookrightarrow \underline{H^i}(\eta_{\mathcal{S}}, \mathbb{C}) \twoheadrightarrow H^{i,0}(\mathcal{S}, \mathbb{C})\}.$$

Since moreover $\mathrm{im}(\theta)$ is a HS, we have $\mathrm{im}(\theta) \subseteq F^1_h\underline{H^i}(\eta_{\mathcal{S}})$; and

$$\mathrm{GHC} \Rightarrow F^1_h\underline{H^i}(\eta_{\mathcal{S}}) \subseteq N^1\underline{H^i}(\eta_{\mathcal{S}}) = 0. \quad \square$$

Lemma 4. (a) *For $\tilde{\mathcal{S}}/\bar{\mathbb{Q}}$ smooth projective of dimension $i + c$, $\exists c$-fold hyperplane section $\mathcal{S}/\bar{\mathbb{Q}}$ s.t. restriction along $\mathcal{S} \overset{\imath}{\hookrightarrow} \tilde{\mathcal{S}}$ induces a well-defined injection $\underline{H^i}(\eta_{\tilde{\mathcal{S}}}) \hookrightarrow \underline{H^i}(\eta_{\mathcal{S}})$.*

(b) *Given $\bar{\tilde{\mathfrak{Z}}} \in Z^p\big((X \times \tilde{\mathcal{S}})_{\bar{\mathbb{Q}}}\big)$ with restriction $\bar{\mathfrak{Z}}$ to $X \times \mathcal{S}$ (and write $\tilde{\mathfrak{Z}}$, $\mathfrak{Z}$ for their resp. restrictions to $X \times \eta_{\tilde{\mathcal{S}}}$, $X \times \eta_{\mathcal{S}}$). If $c_{\mathcal{D}}(\tilde{\mathfrak{Z}})$, $c_{\mathcal{D}}(\mathfrak{Z})$ belong to $\mathcal{L}^i$ of the resp. $\underline{H^{2p}_{\mathcal{D}}}$'s, then $\imath$ induces a (well-defined) injective map (see proof below) under which $[\tilde{\mathfrak{Z}}]_i \mapsto [\mathfrak{Z}]_i$. If they belong to resp. $\mathcal{L}^{i+1}$'s and $[\tilde{\mathfrak{Z}}] = 0$ ($\Rightarrow [\mathfrak{Z}] = 0$), then $\imath$ induces injections sending $[AJ(\tilde{\mathfrak{Z}})]^{\mathrm{tr}}_i \mapsto [AJ(\mathfrak{Z})]^{\mathrm{tr}}_i$, $[AJ(\tilde{\mathfrak{Z}})]^{\mathrm{sf}}_i \mapsto [AJ(\mathfrak{Z})]^{\mathrm{sf}}_i$.*

(c) *Assume* HC. *Then if $c_{\mathcal{D}}(\tilde{\mathfrak{Z}}) \in \mathcal{L}^j\underline{H^{2p}_{\mathcal{D}}}\big(X \times \eta_{\tilde{\mathcal{S}}}, \mathbb{Q}(p)\big)$, one can choose $\bar{\tilde{\mathfrak{Z}}}$ so that (in* (b)) $c_{\mathcal{D}}(\mathfrak{Z}) \in \mathcal{L}^j\underline{H^{2p}_{\mathcal{D}}}\big(X \times \eta_{\mathcal{S}}, \mathbb{Q}(p)\big)$.

Proof. (a) Arguing for $c = 1$, let $\mathcal{S} \subseteq \tilde{\mathcal{S}}$ be a smooth $\bar{\mathbb{Q}}$-hyperplane section and choose resp. codim.-1 $\bar{\mathbb{Q}}$-subvarieties $D \subseteq \mathcal{S}$, $\tilde{D} \subseteq \tilde{\mathcal{S}}$ as follows: D sufficiently "large" that $\underline{H^i}(\mathcal{S} \backslash D) \xrightarrow{\cong} \underline{H^i}(\eta_{\mathcal{S}})$; and $\tilde{D}$ properly intersecting $\mathcal{S}$ with $\tilde{D} \cap \mathcal{S} \supseteq D$. By [AS], Thm.

6.1.1 (a version of affine weak Lefschetz), $H^i(\tilde{\mathscr{S}}\backslash\tilde{D}) \hookrightarrow H^i(\mathscr{S}\backslash\tilde{D}\cap\mathscr{S})$ and so $\underline{H}^i(\tilde{\mathscr{S}}\backslash\tilde{D}) \overset{'\imath^*}{\hookrightarrow} \underline{H}^i(\mathscr{S}\backslash\tilde{D}\cap\mathscr{S})$. Moreover, by [AK], Thm. 1.1(3), $\imath^* : H^i(\tilde{\mathscr{S}}) \hookrightarrow H^i(\mathscr{S})$ respects the coniveau filtration. Hence we get a commutative diagram

$$\begin{tikzcd} H^i(\tilde{\mathscr{S}}) \arrow[r, two heads] \arrow[d, hook, "\imath^*"'] & \dfrac{H^i(\tilde{\mathscr{S}})}{\operatorname{im} H^i_{\tilde{D}}(\tilde{\mathscr{S}})} \arrow[r, two heads] \arrow[d, hook, "'\imath^*"'] & \dfrac{H^i(\tilde{\mathscr{S}})}{N^1 H^i(\tilde{\mathscr{S}})} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "''\imath^*"'] & \underline{H}^i(\eta_{\tilde{\mathscr{S}}}) \\ H^i(\mathscr{S}) \arrow[r, two heads] & \dfrac{H^i(\mathscr{S})}{\operatorname{im} H^i_{\tilde{D}\cap\mathscr{S}}(\mathscr{S})} \arrow[r, "\cong"] & \dfrac{H^i(\mathscr{S})}{N^1 H^i(\mathscr{S})} \arrow[r, leftrightarrow, "\cong"] & \underline{H}^i(\eta_{\mathscr{S}}) \end{tikzcd}$$

from which injectivity of $''\imath^*$ is obvious. (Iterating this procedure proves it for $c > 1$.)

(b) Plug the injection of (a) into $\operatorname{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-p), (\text{—}) \otimes H^{2p-i}(X)\big)$,

$$J^p\left((\text{—}) \otimes \frac{H^{2p-i-1}(X)}{F_h^{p-i} H^{2p-i-1}(X)}\right), \quad \text{and} \quad J^p\left(\frac{(\text{—}) \otimes H^{2p-i-1}(X)}{SF_h^{(1,p-i)}\{\text{num}\}}\right).$$

This automatically yields injections except in the last case, where we need $\dfrac{\underline{H}^i(\eta_{\tilde{\mathscr{S}}}) \otimes H^{2p-i-1}(X)}{SF_h^{(1,p-i)}\{\text{num}\}} \hookrightarrow \dfrac{\underline{H}^i(\eta_{\mathscr{S}}) \otimes H^{2p-i-1}(X)}{SF_h^{(1,p-i)}\{\text{num}\}}$ (in order that the J^p's inject). It suffices to show

$$SF_{h,\mathscr{S}}^{(1,p-i)} \cap \operatorname{im}\{\underline{H}^i(\eta_{\tilde{\mathscr{S}}}) \otimes H^{2p-i-1}(X)\} = SF_{h,\tilde{\mathscr{S}}}^{(1,p-i)}.$$

This follows by describing $SF_h^{(1,p-i)}$ as the largest subHS contained in

$$\ker\{\underline{H}^i(\eta) \otimes H^{2p-i-1}(X) \to \underline{H}^{i,0}(\eta, \mathbb{C}) \otimes H^{p-i-1,p}(X, \mathbb{C})\}$$

and noting that $\underline{H}^{i,0}(\eta_{\tilde{\mathscr{S}}}, \mathbb{C}) \hookrightarrow \underline{H}^{i,0}(\eta_{\mathscr{S}}, \mathbb{C})$.

(c) By [mS], sec. 1.6, we can arrange that $c_{\mathscr{D}}(\bar{\tilde{\mathfrak{Z}}}) \in \mathscr{L}^i_{(X\times\tilde{\mathscr{S}})/\tilde{\mathscr{S}}} H^{2p}_{\mathscr{D}}\big(X \times \tilde{\mathscr{S}}, \mathbb{Q}(p)\big)$, by modifying an initial choice of $\bar{\tilde{\mathfrak{Z}}}$ along $X \times D$ where $D \subseteq \tilde{\mathscr{S}}$ is a divisor defined $/\bar{\mathbb{Q}}$. (This uses the HC.) The conclusion then follows by functoriality of Leray. □

Remark 1. (i) We emphasize that the passage from $\tilde{\mathfrak{Z}} \mapsto \bar{\tilde{\mathfrak{Z}}} \mapsto \bar{\mathfrak{Z}} \mapsto \mathfrak{Z}$ is not a well-defined map of cycles (only $\bar{\tilde{\mathfrak{Z}}} \mapsto \bar{\mathfrak{Z}}$ is). However, (b) says that at least certain *invariants* of $\mathfrak{Z}$ will only depend on those of $\tilde{\mathfrak{Z}}$, and not on the choice of lifting $\tilde{\mathfrak{Z}} \mapsto \bar{\tilde{\mathfrak{Z}}}$.

(ii) Given $\tilde{\mathscr{Z}}$ with (complete) spread $\bar{\tilde{\mathfrak{Z}}}$, $\bar{\mathfrak{Z}}$ should be viewed as a spread of a "specialization" $\mathscr{Z}$ of $\tilde{\mathscr{Z}}$ (defined over a field of lesser transcendence degree $/\bar{\mathbb{Q}}$). Namely, if $p \in \mathscr{S}(\mathbb{C})$ is very general (hence somewhat *less* than very general in $\tilde{\mathscr{S}}(\mathbb{C})$), take $\mathscr{Z}$ to be the restriction of $\bar{\mathfrak{Z}}$ along $X \times \{p\} \hookrightarrow X \times \mathscr{S}$. [Note that this is different (less delicate) than the kind of specialization considered in [GGP], [mS].]

2.5. Change of spread field. We now make a slight extension to the case where X is not defined $/\bar{\mathbb{Q}}$, to be used in Corollary 2 (and, to a lesser extent, Example 3).

Suppose we have $L \subseteqq K(\subseteqq \mathbb{C})$ both f.g. $/\bar{\mathbb{Q}}$, with $\mathrm{trdeg}(K/L) =: t \geqq 1$. Then $\exists \mathscr{S}/\bar{\mathbb{Q}}$ with $s \in \mathscr{S}(\mathbb{C})$ such that $\mathrm{ev}_s : \bar{\mathbb{Q}}(\mathscr{S}) \xrightarrow{\cong} K$, and $\mathscr{M}/\bar{\mathbb{Q}}$ with a morphism $\rho : \mathscr{S} \to \mathscr{M}$ such that $\mathrm{ev}_{\rho(s)} : \bar{\mathbb{Q}}(\mathscr{M}) \xrightarrow{\cong} L$.

Remark 2. If $\mathscr{S}$, $\mathscr{M}$ come from Lemma 2(a), ρ is a priori a rational map, restricting to a morphism only on $U \subseteqq \mathscr{S}$ Zariski open. Take the closure in $\mathscr{S} \times \mathscr{M}$ of $\mathrm{graph}(\rho|_U) \subseteqq U \times \mathscr{M} \subseteqq \mathscr{S} \times \mathscr{M}$ and let $\mathscr{S}'$ be a desingularization of the result. Then one has obvious morphisms $\mathscr{S}' \twoheadrightarrow \mathscr{S}$ and $\mathscr{S}' \to \mathscr{M}$, and the first is a birational equivalence $(\bar{\mathbb{Q}}(\mathscr{S}') \cong \bar{\mathbb{Q}}(\mathscr{S}))$; so just take "$\mathscr{S}$" in the above to be $\mathscr{S}'$.

Now write $\mu = \rho(s) \in \mathscr{M}(\mathbb{C})$ and $\mathscr{T} := \rho^{-1}(\mu) \overset{\imath}{\hookrightarrow} \mathscr{S}_L$, and note that $L(\mathscr{T}) \cong K$ (again via ev_s). Let X be defined $/L$ and $\mathscr{Z} \in Z^p(X_K)$; then one has complete $\bar{\mathbb{Q}}$-spreads $\bar{\mathfrak{Z}} \in Z^p(\bar{\mathfrak{X}}_{\bar{\mathbb{Q}}})$ and $\bar{\mathfrak{X}} \xrightarrow{\pi} \mathscr{S}$, formally restricting to $\mathfrak{X} := \varprojlim \pi^{-1}(\mathscr{U})$ [over $\mathscr{U}_{/\bar{\mathbb{Q}}} \subseteqq \mathscr{S}$ affine Zar. op.] and $\mathfrak{Z} \in Z^p(\mathfrak{X})$. Moreover, one has the *partial* (L-)spreads

$$\bar{\mathfrak{X}}_L \times_{\mathscr{S}_L} \mathscr{T} = X_L \times \mathscr{T} \overset{\imath}{\hookrightarrow} \bar{\mathfrak{X}}_L \quad \text{and} \quad \overline{\mathfrak{Z}_{\mathscr{T}}} := \imath^*(\bar{\mathfrak{Z}}_L) \in Z^p(X_L \times \mathscr{T});$$

we write $\eta_{\mathscr{T}} = \varprojlim \mathscr{V}$ [$\mathscr{V}_{/L} \subseteqq \mathscr{T}$ affine Zar. op.] and note that this is formally the restriction to $\mathscr{T}$ of $(\eta_{\mathscr{S}})_L$.

By functoriality of the Deligne class we get a commuting diagram

$$\begin{tikzcd} CH^p(X_K) \arrow[r, "\cong"] \arrow[dr, "\cong"'] & CH^p(\mathfrak{X}_{\bar{\mathbb{Q}}}) \arrow[r, "c_{\mathscr{D}}"] \arrow[d] & \underline{H}_{\mathscr{D}}^{2p}(\mathfrak{X}_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}(p)) \arrow[d, "\imath^*"] \\ & CH^p((X_L \times \eta_{\mathscr{T}})_L) \arrow[r, "c_{\mathscr{D}}"] & \underline{H}_{\mathscr{D}}^{2p}((X \times \eta_{\mathscr{T}})_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}(p)) \end{tikzcd}$$

where the two $\cong$'s are (resp.) $\bar{\mathbb{Q}}$- and L-spread maps. Writing $\Psi^{K/\bar{\mathbb{Q}}}$ and $\Psi^{K/L}$ for the top and bottom compositions, the Leray filtrations on $\underline{H}_{\mathscr{D}}$ for $\bar{\mathfrak{X}} \xrightarrow{\pi} \mathscr{S}$ and $X \times \mathscr{T} \xrightarrow{\pi_{\mathscr{T}}} \mathscr{T}$ induce (via the two Ψ's) filtrations $\mathscr{L}^{\bullet}_{K/\bar{\mathbb{Q}}} \subseteqq \mathscr{L}^{\bullet}_{K/L}$ on $CH^p(X_K)$. (That $\imath^*(\mathscr{L}^{\bullet}_{\pi}) \subseteqq \mathscr{L}^{\bullet}_{\pi_{\mathscr{T}}}$ follows from Lewis's explicit description of $\mathscr{L}^{\bullet}$ on $\underline{H}^*_{\mathscr{D}}$ on the level of representative Deligne-homology cochains.) Hence we have maps

$$\Psi_i^{K/\bar{\mathbb{Q}}} : \mathscr{L}^i_{(K/\bar{\mathbb{Q}})} CH^p(X_K) \to Gr^i_{\mathscr{L}_\pi} \underline{H}_{\mathscr{D}}^{2p}(\mathfrak{X}, \mathbb{Q}(p)),$$
$$\Psi_i^{K/L} : \mathscr{L}^i_{K/L} CH^p(X_K) \to Gr^i_{\mathscr{L}_{\pi_{\mathscr{T}}}} \underline{H}_{\mathscr{D}}^{2p}(X \times \eta_{\mathscr{T}}, \mathbb{Q}(p)).$$

The former extends the $\Psi_i^{K/\bar{\mathbb{Q}}}$ defined [for the case $\mathfrak{X} = X \times \eta_{\mathscr{S}}$] in §2.4 above, but is difficult to compute; the latter is easy to compute with (2), (3), (4), and $\Psi_i^{K/L} = \imath^* \circ \Psi_i^{K/\bar{\mathbb{Q}}}$ on $\langle \mathscr{Z} \rangle \in \mathscr{L}^i_{K/\bar{\mathbb{Q}}}$. We state what we will use:

Lemma 5. *If $\langle \mathscr{Z} \rangle \in \mathscr{L}^i_{K/L} CH^p(X_K)$ and $\Psi_i^{K/L}(\mathscr{Z}) \neq 0$, then $\mathscr{Z} \overset{\mathrm{rat}}{\not\equiv} 0$. More precisely, one of two things is true*:

(i) $\langle \mathscr{Z} \rangle \notin \mathscr{L}^i_{K/\bar{\mathbb{Q}}}$,

(ii) $\langle \mathscr{Z} \rangle \in \mathscr{L}^i_{K/\bar{\mathbb{Q}}}$ *and* $\Psi_i^{K/\bar{\mathbb{Q}}}(\mathscr{Z}) \neq 0$.

Remark 3. For Example 3 it will also be helpful to note that $\Psi_i^{K/\bar{\mathbb{Q}}}(\mathscr{Z})$ resp. $\Psi_i^{K/L}(\mathscr{Z})$ "split" as before into $[\mathfrak{Z}]_i$ and $[AJ(\mathfrak{Z})]_{i-1}$, resp. $[\mathfrak{Z}_{\mathscr{T}}]_i$ and $[AJ(\mathfrak{Z}_{\mathscr{T}})]_{i-1}$, with e.g. $[\mathfrak{Z}]_i \in \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-p), \underline{H}^i(\eta_{\mathscr{S}}, R^{2p-i}\pi_*\mathbb{Q})\big)$ being sent to $[\mathfrak{Z}_{\mathscr{T}}]_i$ by $\imath^*$.

2.6. Exterior products of cycles respect the filtration. In the more general context where X may not be defined $/\bar{\mathbb{Q}}$, and for cycles of any codimension, it is proved in [L1] that on $CH^*(X_{\mathbb{C}})$, $\mathscr{L}^i \cdot \mathscr{L}^j \subseteq \mathscr{L}^{i+j}$ under the intersection product. Moreover, push-forwards and pullbacks preserve $\mathscr{L}^\bullet$. This leads immediately to the following.

Lemma 6. *Given $\langle \mathscr{Z}_i \rangle \in \mathscr{L}^{j_i}_{K_i/\bar{\mathbb{Q}}} CH_0\big((Y_i)_{K_i}\big)$ for $i = 1, 2$, we have*

$$\langle \mathscr{Z}_1 \times \mathscr{Z}_2 \rangle \in \mathscr{L}^{j_1+j_2}_{K/\bar{\mathbb{Q}}} CH_0(X_K) \quad \textit{where } K := \bar{\mathbb{Q}}(K_1, K_2) \textit{ and } X = Y_1 \times Y_2.$$

Proof. Writing $\pi_i : X \twoheadrightarrow Y_i$ $(i = 1, 2)$, $\mathscr{Z} = (\pi_1^*\mathscr{Z}_1) \cdot (\pi_2^*\mathscr{Z}_2)$ and we use the 2 properties of $\mathscr{L}^\bullet$ just mentioned. $\square$

Remark 4. We emphasize that the Y_i need not be defined $/\bar{\mathbb{Q}}$, and that K_1, K_2 need not be "algebraically independent" in the sense of Lemma 1(b); they could even be the same field (so that $K = K_1 = K_2$).

3. General × special

In this section we present various results and examples which are all variations on the following theme: that the product of a 0-cycle which is "general" in some appropriate sense by a "special" 0-cycle with nontrivial Albanese image, is not rationally equivalent to zero.

Theorem 1. *Consider Y_1 and Y_2 smooth projective varieties $/\bar{\mathbb{Q}}$ with resp. dimensions d_1 and d_2. Let $K \subseteq \mathbb{C}$ be f.g. $/\bar{\mathbb{Q}}$ of* trdeg. *j, and $\langle \mathscr{V} \rangle \in \mathscr{L}^j CH_0\big((Y_1)_K\big)$ be such that its complete spread $\bar{\mathfrak{V}} \in Z^{d_1}\big((\mathscr{S} \times Y_1)_{\bar{\mathbb{Q}}}\big)$ induces a nontrivial map $\Omega^j(Y_1) \to \Omega^j(\mathscr{S})$ of holomorphic forms. Take $\langle \mathscr{W} \rangle \in CH_0^{\hom}\big((Y_2)_{\bar{\mathbb{Q}}}\big)$ with $0 \neq AJ_{Y_2}(\mathscr{W}) \in J^{d_2}(Y_2) := J^{d_2}\big(H^{2d_2-1}(Y_2)\big)$.*

Then $\langle \mathscr{Z} := \mathscr{V} \times \mathscr{W} \rangle \in CH_0\big((Y_1 \times Y_2)_K\big)$ is not zero. More precisely,

$$\langle \mathscr{Z} \rangle \in \mathscr{L}^{j+1} CH_0 \quad \textit{and} \quad [\mathfrak{Z}] = 0;$$

but $[AJ(\mathfrak{Z})]_j \neq 0$ (equiv. $AJ^j_{Y_1 \times Y_2}(\mathscr{Z}) \neq 0$), hence $\langle \mathscr{Z} \rangle \notin \mathscr{L}^{j+2}$.

Proof. The fundamental class $[\bar{\mathfrak{V}}]$ has j^{th} Künneth component

$$[\bar{\mathfrak{V}}]_j \in H^j(\mathscr{S}) \otimes H^{2d_1-j}(Y_1),$$

which also lies in the r.h.s. of

$$\{H^j(\mathscr{S}, \mathbb{C}) \otimes H^{2d_1-j}(Y_1, \mathbb{C})\}^{(d_1,d_1)} = \bigoplus_{i=0}^{j} H^{i,j-i}(\mathscr{S}, \mathbb{C}) \otimes H^{d_1-i,d_1+i-j}(Y_1, \mathbb{C}).$$

We may write uniquely $[\bar{\mathfrak{V}}]_j = \sum_{i=1}^{j} [\bar{\mathfrak{V}}]_{(i,j-i)}$ and $[\bar{\mathfrak{V}}]_{(j,0)} = \sum_{\ell} \alpha_\ell \otimes \nu_\ell \big(= \overline{[\bar{\mathfrak{V}}]_{(0,j)}}\big)$ where $\{\alpha_\ell\}$

is a basis of $H^{j,0}(\mathscr{S},\mathbb{C})$ and $v_\ell \in H^{d_1-j,d_1}(Y_1,\mathbb{C})$. Our hypothesis on $\bar{\mathfrak{V}}$ implies at least that one v_ℓ, say v_1, is nonzero.

Set $d = d_1 + d_2$, $X = Y_1 \times Y_2$. Since $\mathscr{W}$ is defined $/\bar{\mathbb{Q}}$, the $\bar{\mathbb{Q}}$-spread of $\mathscr{Z}$ is the restriction $\mathfrak{Z}$ of

$$\bar{\mathfrak{Z}} = \bar{\mathfrak{V}} \times \mathscr{W} \in Z^d\big((\mathscr{S} \times X)_{\bar{\mathbb{Q}}}\big)$$

to $\eta_{\mathscr{S}} \times Y_1 \times Y_2$. Already $\bar{\mathfrak{Z}} \overset{\hom}{\equiv} 0$ with bounding chain

$$\partial^{-1}\bar{\mathfrak{Z}} := \bar{\mathfrak{V}} \times \partial^{-1}\mathscr{W} \tag{5}$$

of real dimension $2j+1$. (Here $\partial^{-1}\mathscr{W}$ is a real 1-chain bounding on $\mathscr{W}$ which is *fixed* for the remainder of the proof.) Since $\langle \mathscr{V} \rangle \in \mathscr{L}^j$, $[\bar{\mathfrak{V}}]_i \mapsto 0 \in \underline{H}^i(\eta_{\mathscr{S}}) \otimes H^{2d-i}(Y_1)$ for all $i < j$. Together with (5) this implies $\left(\int_{\partial^{-1}\bar{\mathfrak{Z}}} (\cdot) \right) \mapsto 0 \in J^d\big(\underline{H}^i(\eta_{\mathscr{S}}) \otimes H^{2d-i-1}(X)\big)$, hence $[AJ(\mathfrak{Z})]_i = 0$ $(\forall i < j)$ and $\langle \mathscr{Z} \rangle \in \mathscr{L}^{j+1}$.

Let $[\bar{\mathfrak{V}}]_j^\vee \in H^j(\mathscr{S}) \otimes H^j(Y_1)$ be any rational type (j,j)-class dual to $[\bar{\mathfrak{V}}]_j$; that is, under [the restriction of] the Poincaré duality pairing $H^{2d_1}(\mathscr{S} \times Y_1) \otimes H^{2j}(\mathscr{S} \times Y_1) \to \mathbb{Q}$, $[\bar{\mathfrak{V}}]_j \otimes [\bar{\mathfrak{V}}]_j^\vee \mapsto 1$. Denote by $\operatorname{ann}([\bar{\mathfrak{V}}]_j) \subseteq H^j(\mathscr{S}) \otimes H^j(Y_1)$ the subHS annihilated by $[\bar{\mathfrak{V}}]_j$. Consider the following relative dual pairs of HS's:

$$\mathscr{H}_0 = H^j(\mathscr{S}) \otimes H^{2d-j-1}(X) \supseteq H^j(\mathscr{S}) \otimes H^{2d_1-j}(Y_1) \otimes H^{2d_2-1}(Y_2) = \mathscr{H}_1,$$
$$\mathscr{K}_0 = H^j(\mathscr{S}) \otimes H^{j+1}(X) \supseteq H^j(\mathscr{S}) \otimes H^j(Y_1) \otimes H^1(Y_2) = \mathscr{K}_1;$$

and

$$\mathscr{H}_{\mathfrak{V}} = \mathbb{Q}[\bar{\mathfrak{V}}]_j \otimes H^{2d_2-1}(Y_2) \subseteq \mathscr{H}_1, \quad \mathscr{K}_{\mathfrak{V}} = \mathbb{Q}([\bar{\mathfrak{V}}]_j^\vee) \otimes H^1(Y_2) \subseteq \mathscr{K}_1.$$

(The $\mathscr{H}$'s are of weight $2d-1$, the $\mathscr{K}$'s of weight $2j+1$; so here d, $d+j$ replace the n, d (resp.) of §§2.1–2.) If we take

$$\tilde{\Xi} = \left(\int_{\partial^{-1}\bar{\mathfrak{Z}}} (\cdot) \right) \in \{F^{j+1}\mathscr{K}_0^{\mathbb{C}}\}^\vee \quad \text{and} \quad \tilde{\xi} = [\bar{\mathfrak{V}}]_j \otimes \left(\int_{\partial^{-1}\mathscr{W}} (\cdot) \right) \in \{F^{j+1}\mathscr{K}_{\mathfrak{V}}^{\mathbb{C}}\}^\vee$$

then clearly the former annihilates $F^{j+1}\mathscr{K}_2 = F^{j+1}\{\operatorname{ann}([\bar{\mathfrak{V}}]_j)_{\mathbb{C}} \otimes H^1(Y_2,\mathbb{C})\}$ completely while agreeing with $\tilde{\xi}$ on $F^{j+1}\mathscr{K}_{\mathfrak{V}}^{\mathbb{C}} = \mathbb{C}[\bar{\mathfrak{V}}]_j^\vee \otimes H^{1,0}(Y_2,\mathbb{C})$. Note that $0 \neq \xi = [\bar{\mathfrak{V}}]_j \otimes AJ_{Y_2}(\mathscr{W}) \in \mathbb{C}[\bar{\mathfrak{V}}]_j \otimes J^{d_2}(Y_2) \cong J^d(\mathscr{H}_{\mathfrak{V}})$ by assumption.

Now recall the notation $\beta_0 : J(\mathscr{H}_0) \twoheadrightarrow J(\mathscr{H}_0/\mathscr{G}_0)$ from Lemma 1. Let

$$\mathscr{G}_0 = N^1 H^j(\mathscr{S}) \otimes H^{2d-j-1}(X) + H^j(\mathscr{S}) \otimes F_h^{d-j} H^{2d-j-1}(X) \subseteq \mathscr{H}_0,$$

so that $[AJ(\mathfrak{Z})]_j = \beta_0(\Xi) \in J^d(\mathscr{H}_0/\mathscr{G}_0) = J^d\left(\underline{H}^j(\eta_{\mathscr{S}}) \otimes \widetilde{\dfrac{H^{2d-j-1}(X)}{F_h^{d-j}}} \right)$. We must check that the image of $\mathscr{G}_0$ under the Künneth projection $\widetilde{\mathrm{pr}}_1 : \mathscr{H}_0 \twoheadrightarrow \mathscr{H}_1$ is contained in $\mathscr{H}_1 \cap \mathscr{G}_0$. Set

$$\mathscr{N} = N^1 H^j(\mathscr{S}) \otimes H^{2d_1-j}(Y_1) \otimes H^{2d_2-1}(Y_2),$$
$$\mathscr{F} = H^j(\mathscr{S}) \otimes F_h^{d-j}\{H^{2d_1-j}(Y_1) \otimes H^{2d_2-1}(Y_2)\}.$$

Since $\theta : H^{2d-j-1}(X) \twoheadrightarrow H^{2d_1-j}(Y_1) \otimes H^{2d_2-1}(Y_2)$ is a morphism of HS,

$$\begin{aligned}\theta\big(F_h^{d-j} H^{2d-j-1}(X)\big) &\subseteqq F_h^{d-j}\{H^{2d_1-j}(Y_1) \otimes H^{2d_2-1}(Y_2)\} \\ &\subseteqq \{H^{2d_1-j}(Y_1) \otimes H^{2d_2-1}(Y_2)\} \cap F_h^{d-j} H^{2d-j-1}(X) \\ &\subseteqq \theta\big(F_h^{d-j} H^{2d-j-1}(X)\big).\end{aligned}$$

It follows that $\widetilde{\mathrm{pr}_1}(\mathscr{G}_0) \subseteqq \mathscr{N} + \mathscr{F} \subseteqq \mathscr{H}_1 \cap \mathscr{G}_0 \subseteqq \widetilde{\mathrm{pr}_1}(\mathscr{G}_0)$, which gives us what we want but also that $\mathscr{H}_1 \cap \mathscr{G}_0 = \mathscr{N} + \mathscr{F}$.

Now by Lemma 1(i) we are done (i.e., $\beta_0(\Xi) \neq 0$) if we can show that $\mathscr{H}_{\mathfrak{V}} \cap (\mathscr{H}_1 \cap \mathscr{G}_0)$ is zero in $\mathscr{H}_1$.

Take $\mathrm{p} : H^j(\mathscr{S}, \mathbb{C}) \twoheadrightarrow H^{j,0}(\mathscr{S}, \mathbb{C}) \oplus H^{0,j}(\mathscr{S}, \mathbb{C})$ to be the projection with kernel $H^{j-1,1}(\mathscr{S}, \mathbb{C}) \oplus \cdots \oplus H^{1,j-1}(\mathscr{S}, \mathbb{C}) \supseteqq N^1 H^j(\mathscr{S}, \mathbb{C})$. Then the induced

$$\mathrm{p}_1 : \mathscr{H}_1^{\mathbb{C}} \twoheadrightarrow \{H^{j,0}(\mathscr{S}, \mathbb{C}) \oplus H^{0,j}(\mathscr{S}, \mathbb{C})\} \otimes H^{2d_1-j}(Y_1, \mathbb{C}) \otimes H^{2d_2-1}(Y_2, \mathbb{C})$$

kills $\mathscr{N}$.

Next let $\underline{\mathrm{n}}$, $\underline{\mathrm{f}}$, $[\overline{\mathfrak{V}}]_j \otimes \Gamma$ be arbitrary elements of $\mathscr{N}_{\mathbb{C}}$, $\mathscr{F}_{\mathbb{C}}$, and $\mathscr{H}_{\mathfrak{V}}^{\mathbb{C}}$ ($\Gamma \in H^{2d_2-1}(Y_2, \mathbb{C})$ arbitrary), and suppose $\underline{\mathrm{n}} + \underline{\mathrm{f}} = [\overline{\mathfrak{V}}]_j \otimes \Gamma$ in $\mathscr{H}_1^{\mathbb{C}}$. In order to prove $\mathscr{H}_{\mathfrak{V}}^{\mathbb{C}} \cap (\mathscr{N}_{\mathbb{C}} + \mathscr{F}_{\mathbb{C}}) = \{0\}$, we must show $\Gamma = 0$. Apply p_1 to both sides to get $\mathrm{p}_1(\underline{\mathrm{f}}) = \mathrm{p}_1([\overline{\mathfrak{V}}]_j \otimes \Gamma)$, i.e.

$$\sum_\ell \alpha_\ell \otimes A_\ell + \sum_\ell \overline{\alpha_\ell} \otimes B_\ell = \sum_\ell \alpha_\ell \otimes v_\ell \otimes \Gamma + \sum_\ell \overline{\alpha_\ell} \otimes \overline{v_\ell} \otimes \Gamma$$

for unique classes $A_j, B_j \in \big(F_h^{d-j}\{H^{2d_1-j}(Y_1) \otimes H^{2d_2-1}(Y_2)\}\big) \otimes \mathbb{C}$. Hence we must have $v_1 \otimes \Gamma = A_1$, $\overline{v_1} \otimes \Gamma = B_1$ in $\mathsf{H}_{\mathbb{C}} := H^{2d_1-j}(Y_1, \mathbb{C}) \otimes H^{2d_2-1}(Y_2, \mathbb{C})$. That is, $v_1 \otimes \Gamma$ and $\overline{v_1} \otimes \Gamma$ must belong to (a subspace of)

$$\mathsf{H}_{\mathbb{C}}^{d-j,d-1} \oplus \cdots \oplus \mathsf{H}_{\mathbb{C}}^{d-1,d-j}.$$

Since v_1 is nonzero of pure type $(d_1 - j, d_1)$, Γ belongs to

$$H^{d_2,d_2-1}(Y_2, \mathbb{C}) \oplus [H^{d_2+1,d_2-2}(Y_2, \mathbb{C}) \oplus \cdots \oplus H^{d_2+j-1,d_2-j}(Y_2, \mathbb{C})]$$

(bracketed terms $= 0$); but since $\overline{v_1}$ is of type $(d_1, d_1 - j)$, Γ is in

$$[H^{d_2-j,d_2+j-1}(Y_2, \mathbb{C}) \oplus \cdots \oplus H^{d_2-2,d_2+1}(Y_2, \mathbb{C})] \oplus H^{d_2-1,d_2}(Y_2, \mathbb{C}).$$

Hence $\Gamma = 0$. $\square$

Here is one way to construct $\mathscr{V}$ on Y_1 with the properties assumed in the statement of Theorem 1. (Note that such a $\mathscr{V}$ automatically has $cl^j_{Y_1}(\mathscr{V}) \neq 0$ and $\mathscr{V} \underset{\mathrm{rat}}{\not\equiv} 0$.)

Consider some j-dimensional (possibly singular) subvariety $S_{/\bar{\mathbb{Q}}} \subseteqq Y_1$ with desingularization $\mathscr{S} \xrightarrow{\iota} Y_1$, such that the restriction (ι^*) induces a nontrivial map of holomorphic j-forms. Let $p_0 \in \mathscr{S}(\mathbb{C})$ be very general, and write p for its image in Y_1. An obvious choice of complete spread for p is the graph of ι, $\bar{\mathfrak{p}} := (\mathrm{id} \times \iota)_*(\Delta_{\mathscr{S}}) \in Z^{d_1}(\mathscr{S} \times Y_1)$, whose action $\bar{\mathfrak{p}}^* : H^j(Y_1) \to H^j(\mathscr{S})$ is of course ι^*. Suppose $[\Delta_{\mathscr{S}}] \in H^j(\mathscr{S} \times \mathscr{S})$ has algebraic Künneth components $\Delta_{\mathscr{S}}(i, 2j-i) \in Z^j\big((\mathscr{S} \times \mathscr{S})_{\bar{\mathbb{Q}}}\big)$ with $[\Delta_{\mathscr{S}}]_i = [\Delta_{\mathscr{S}}(i, 2j-i)]$, as is the case if $\mathscr{S}$ is a curve, surface, abelian variety, smooth complete intersection (in $\mathbb{P}^N$), or arbitrary product of these. Then $\langle \Delta_{\mathscr{S}}(j,j)_* p_0 \rangle \in \mathscr{L}^j CH_0(\mathscr{S}_{\mathbb{C}})$, and $\mathscr{L}^\bullet$ is preserved under ι_* (see [L1]); hence the class of $\mathscr{V} := \iota_*\big(\Delta_{\mathscr{S}}(j,j)_* p_0\big)$ lies in $\mathscr{L}^j CH_0\big((Y_1)_{\mathbb{C}}\big)$. A complete spread for $\mathscr{V}$ is $\bar{\mathfrak{V}} := (\mathrm{id} \times \iota)_* \Delta_{\mathscr{S}}(j,j) \in Z^{d_1}(\mathscr{S} \times Y_1)$, so that

$$\bar{\mathfrak{V}}^* = \Delta_{\mathscr{S}}(j,j)^* \circ \iota^* = (\delta_{ij} \cdot \text{identity}) \circ \iota^* : H^i(Y_1) \to H^i(\mathscr{S}).$$

From the hypothesis on ι it follows that $\bar{\mathfrak{V}}^* : \Omega^j(Y_1) \to \Omega^j(\mathscr{S})$ is nonzero; so $\mathscr{V}$ has the desired properties. Here are two concrete instances:

Example 1. If $Y_1 \subseteqq \mathbb{P}^N$ is a (smooth) complete intersection with $\deg(K_{Y_1}) \geqq 0$, take $j = d_1$ and $\mathscr{S} = S = Y_1$. The construction gives $\mathscr{V} = p - o$, where $p \in Y_1(\mathbb{C})$ is very general and $o \in Y_1(\bar{\mathbb{Q}})$.

Example 2. Let $X = \mathscr{C}_1 \times \cdots \times \mathscr{C}_n$ be a product of curves defined $/\bar{\mathbb{Q}}$ each of genus $\geqq 1$. On each $\mathscr{C}_i$ let $\mathscr{W}_i$ be a divisor defined $/\bar{\mathbb{Q}}$ with $AJ(\mathscr{W}_i) \neq 0$ in $J^1(\mathscr{C}_i)$, $p_i \in \mathscr{C}_i(\mathbb{C})$ be very general, $o_i \in \mathscr{C}_i(\bar{\mathbb{Q}})$. Assume $\{p_1, \ldots, p_n\}$ "algebraically independent" in the sense that $p_1 \times \cdots \times p_n \in X(\mathbb{C})$ is very general (not defined over a field of trdeg $< n$). Then for each $j \geqq 1$, take

$$S = \mathscr{C}_1 \times \cdots \times \mathscr{C}_j \times \{o_{j+2}\} \times \cdots \times \{o_n\} \subseteqq \mathscr{C}_1 \times \cdots \times \mathscr{C}_j \times \mathscr{C}_{j+2} \times \cdots \times \mathscr{C}_n = Y_1,$$

$p_0 = p_1 \times \cdots \times p_j (\times o_{j+2} \times \cdots \times o_n) \in S(\mathbb{C})$. One obtains from the above construction

$$\langle \mathscr{V} := (p_1 - o_1) \times \cdots \times (p_j - o_j) \times o_{j+2} \times \cdots \times o_n \rangle \in \mathscr{L}^j CH_0\big((Y_1)_{\mathbb{C}}\big)$$

with the properties assumed in the theorem. Hence

$$\langle \mathscr{Z} := (p_1 - o_1) \times \cdots \times (p_j - o_j) \times \mathscr{W}_{j+1} \times o_{j+2} \times \cdots \times o_n \rangle \in \mathscr{L}^{j+1} CH_0(X_{\mathbb{C}})$$

has $AJ_X^j(\mathscr{Z}) \neq 0$, and so $\mathscr{Z} \overset{\text{rat}}{\not\equiv} 0$.

In [K1], this example is tied to Bloch's results ([B1]) for 0-cycles on Abelian varieties (which are dominated by these products of curves). One also gets applications to Calabi-Yau 3-folds.

Here is a slight generalization of Theorem 1 that gives some initial evidence that our invariants are well-behaved under products.

Corollary 1. *Let Y_1, Y_2, $\mathscr{W}$ be as in Theorem* 1 *and take $\langle \tilde{\mathscr{V}} \rangle \in \mathscr{L}^j CH_0\big((Y_1)_{\mathbb{C}}\big)$ with $cl^j_{Y_1}(\tilde{\mathscr{V}}) \neq 0$. Then if the* GHC *holds, $AJ^j_X(\tilde{\mathscr{V}} \times \mathscr{W}) \neq 0$ hence $\tilde{\mathscr{V}} \times \mathscr{W} \overset{\text{rat}}{\not\equiv} 0$.*

Proof. We have $\tilde{K}$ finitely generated $/\bar{\mathbb{Q}}$ and $\tilde{\mathscr{S}}$ (defined $/\bar{\mathbb{Q}}$), such that $\tilde{\mathscr{V}}/\tilde{K}$ and $\bar{\mathbb{Q}}(\tilde{\mathscr{S}}) \cong \tilde{K}$. By definition of $cl^j_{Y_1}$, $0 \neq [\tilde{\mathfrak{V}}]_j \in \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-d_1), \underline{H}^j(\eta_{\tilde{\mathscr{S}}}) \otimes H^{2d_1-j}(Y_1)\big)$.

By Lemma 4(b), (c) (and HC), $\exists$ a j-dimensional section $\mathscr{S}_{/\bar{\mathbb{Q}}} \overset{\iota}{\hookrightarrow} \tilde{\mathscr{S}}$ such that (for $\bar{\mathfrak{V}} = \iota^* \bar{\tilde{\mathfrak{V}}}$) $[\mathfrak{V}]_j = \iota^*[\tilde{\mathfrak{V}}]_j \neq 0$. By GHC$(1, j, \mathscr{S})$ and Lemma 3, $\bar{\mathfrak{V}}^* : \Omega^j(Y_1) \to \Omega^j(\mathscr{S})$ is nontrivial hence $[AJ(\mathfrak{V} \times \mathscr{W})]_j \neq 0$ by the theorem. Now obviously $\tilde{\mathscr{V}} \times \mathscr{W} \in \mathscr{L}^{j+1}$ and $[\bar{\tilde{\mathfrak{V}}} \times \mathscr{W}] = 0$, hence $[AJ(\tilde{\mathfrak{V}} \times \mathscr{W})]_j$ is defined and maps to $[AJ(\mathfrak{V} \times \mathscr{W})]_j$ under ι^*. □

In the theorem and corollary Y_1, Y_2, $\mathscr{W}$ are defined $/\bar{\mathbb{Q}}$, and we would like to have a more flexible statement. One idea is to allow Y_1 (like $\mathscr{V}$) to be defined $/K$, in such a way that the generalized $[\mathfrak{V}]_j$ of Remark 3 is still $\neq 0$. However this turns out to be too optimistic as we now show.

Example 3 (Bielliptic cycle). Let $E_\lambda := \overline{\{y^2 = x(x-1)(x-\lambda)\}} \overset{\mathsf{X}_\lambda}{\to} \mathbb{P}^1_{[x]}$; these form a family $\mathscr{E} \overset{\pi}{\to} \mathscr{M} \subseteq \mathbb{P}^1_{[\lambda]}$. Writing $o_\lambda := (0,0)$, pick $\lambda_2 \in \bar{\mathbb{Q}}$ and $q_{\lambda_2} \in E_{\lambda_2}(\bar{\mathbb{Q}})$ such that $AJ(q_{\lambda_2} - o_{\lambda_2}) \in J^1(E_{\lambda_2})$ is nonzero. Put $\varepsilon := \mathsf{X}_{\lambda_2}(q_{\lambda_2}) \in \mathbb{P}^1(\bar{\mathbb{Q}})$ and choose $q_{\tilde{\lambda}} \in \mathsf{X}^{-1}_{\tilde{\lambda}}(\varepsilon)$ (with ε fixed) continuously in $\lambda \in \mathbb{C}$, which requires lifting to a double cover $\tilde{\mathscr{E}} \to \tilde{\mathscr{M}}$ of the family.

Define $\langle \mathscr{Z}_\lambda := q_\lambda - o_\lambda \rangle \in CH_0(E_\lambda)$, take $\lambda_1 \in \mathbb{C} \backslash \bar{\mathbb{Q}}$ and set $K := \bar{\mathbb{Q}}(\lambda_1)$. The cycle $\mathscr{Z}_{\lambda_1}$ on E_{λ_1} is defined over a quadratic extension of K; its spread $\mathfrak{Z}_1$ on $\tilde{\mathscr{E}}$ yields a normal function $\nu \in \Gamma(\tilde{\mathscr{M}}, \mathscr{J}^1_{\tilde{\mathscr{E}}/\tilde{\mathscr{M}}})$ defined by $\nu(\lambda) := AJ_{E_\lambda}(\mathscr{Z}_\lambda)$. One easily sees this is nontrivial (since if we did not work $\otimes \mathbb{Q}$, it is 2-torsion at $\lambda = \varepsilon$ but not elsewhere); arguing by monodromy, its infinitesimal invariant $\delta\nu \in \Gamma\left(\tilde{\mathscr{M}}, \dfrac{\Omega^1_{\tilde{\mathscr{M}}} \otimes \mathscr{H}^1_{\tilde{\mathscr{E}}/\tilde{\mathscr{M}}}}{\nabla(\mathscr{O}_{\tilde{\mathscr{M}}} \otimes \mathscr{F}^1 \mathscr{H}^1_{\tilde{\mathscr{E}}/\tilde{\mathscr{M}}})}\right) \cong \Gamma(\tilde{\mathscr{M}}, \Omega^1_{\tilde{\mathscr{M}}} \otimes \mathscr{H}^{1,0}_{\tilde{\mathscr{E}}/\tilde{\mathscr{M}}})$ must be nonzero.

According to Remark 3, the generalized $\Psi_1^{K/\bar{\mathbb{Q}}}(\mathscr{Z}_{\lambda_1})$ invariant maps to a generalized $[\mathfrak{Z}_1]_1 \in \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-1), \underline{H}^1(\eta_{\tilde{\mathscr{M}}}, R^1\pi_*\mathbb{Q})\big)$ which itself maps (injectively) to the infinitesimal invariant, hence $[\mathfrak{Z}_1]_1 \neq 0$. So in the notation of Theorem 1, $\mathscr{Z}_{\lambda_1}$ plays the role of $\mathscr{V}$ (in a generalized sense), and $\mathscr{Z}_{\lambda_2}$ (defined $/\bar{\mathbb{Q}}$, with nontrivial AJ image) that of $\mathscr{W}$.

But $\mathscr{Z}_{\lambda_1} \times \mathscr{Z}_{\lambda_2} = (q_{\lambda_1} - o_{\lambda_1}) \times (q_{\lambda_2} - o_{\lambda_2}) \overset{\text{rat}}{\equiv} 0$ (modulo 2-torsion if we did not work $\otimes \mathbb{Q}$). This is because $(q_{\lambda_1}, q_{\lambda_2})$ is a (very general) point on the image of the bielliptic curve $E_{\lambda_1} \times_{\mathbb{P}^1_{[x]}} E_{\lambda_2} =: B$ in $E_1 \times E_2$. Writing σ for $(-\mathrm{id})$ on each E_i, $\sigma \times \sigma$ induces the hyperelliptic involution on B; hence an explicit function on B gives

$$-(q_{\lambda_1}, q_{\lambda_2}) - \big(\sigma(q_{\lambda_1}), \sigma(q_{\lambda_2})\big) + 2(o_{\lambda_1}, o_{\lambda_2}) \overset{\text{rat}}{\equiv} 0.$$

One now easily shows $-(\sigma \times \sigma)(\mathscr{Z}_{\lambda_1} \times \mathscr{Z}_{\lambda_2}) \overset{\text{rat}}{\equiv} \mathscr{Z}_{\lambda_1} \times \mathscr{Z}_{\lambda_2} \overset{\text{rat}}{\equiv} (\sigma \times \sigma)(\mathscr{Z}_{\lambda_1} \times \mathscr{Z}_{\lambda_2})$; the assertion follows.

Here, then, is the appropriate generalization.

Corollary 2. *Let $K \supseteq L$ be an extension (of* trdeg $\geqq j$*) of subfields of $\mathbb{C}$ f.g. $/\bar{\mathbb{Q}}$. Let Y_1, Y_2, $\mathscr{W}$ be defined $/\underline{L}$ but otherwise as in Theorem 1. Referring to §2.5, take $\langle \mathscr{V} \rangle \in \mathscr{L}^j_{K/\bar{\mathbb{Q}}} CH_0\big((Y_1)_K\big)$ s.t. $\overline{\mathfrak{V}_{\mathscr{T}}} \in Z^{d_1}\big((\mathscr{T} \times Y_1)_L\big)$ induces $\Omega^j(Y_1) \to \Omega^j(\mathscr{T})$ nontrivial.*

Then $\mathscr{Z} := \mathscr{V} \times \mathscr{W} \overset{\text{rat}}{\not\equiv} 0$; more precisely, $\Psi^{K/\bar{\mathbb{Q}}}_{j+1}(\mathscr{Z}) \neq 0$.

Proof. By the proof of Theorem 1, we have

$$0 \neq [AJ(\mathfrak{Z}_{\mathscr{T}})]_j^{\mathrm{tr}} \in J^d\left(\underline{H}^j(\eta_{\mathscr{T}}) \otimes \frac{H^{2d-j-1}(X)}{F_h^{d-j}}\right).$$

Hence, $\Psi_{j+1}^{K/L}(\mathscr{Z}) \neq 0$. Since $\mathscr{V} \in \mathscr{L}^j_{K/\bar{\mathbb{Q}}}$ and $\mathscr{W} \in \mathscr{L}^1_{K/\bar{\mathbb{Q}}}$, $\mathscr{Z} \in \mathscr{L}^{j+1}_{K/\bar{\mathbb{Q}}}$ by Lemma 6; hence by Lemma 5, $\Psi_{j+1}^{K/\bar{\mathbb{Q}}}(\mathscr{Z}) \neq 0$. $\square$

Remark 5. We expect no better, in the sense that there are situations where $[\mathfrak{Z}_{\mathscr{T}}]_{j+1} = 0$, $[AJ(\mathfrak{Z}_{\mathscr{T}})]_j \neq 0$ for the *partial* spread but $[\mathfrak{Z}]_{j+1} \neq 0$ for the $\bar{\mathbb{Q}}$-spread. (See [K1], sec. 7.1.)

4. General × general

As far as taking products of cycles that *both* spread is concerned, the easiest case is where each has a nontrivial higher cycle-class. Let Y_1, Y_2 be defined $/\bar{\mathbb{Q}}$. If $\langle \mathscr{V}_i \rangle \in \mathscr{L}^{j_i} CH_0\big((Y_i)_{K_i}\big)$ (for $i = 1,2$) have $cl^{j_i}_{Y_i}(\mathscr{V}_i) \neq 0$, then with "reasonable" assumptions we can expect $cl^{j_1+j_2}_{Y_1\times Y_2}(\mathscr{V}_1 \times \mathscr{V}_2) \neq 0$ (clearly $\langle \mathscr{V}_1 \times \mathscr{V}_2 \rangle \in \mathscr{L}^{j_1+j_2}$ by Lemma 6). Namely, we need a hypothesis that guarantees the spread of $\mathscr{V}_1 \times \mathscr{V}_2$ to be just $\bar{\mathfrak{V}}_1 \times \bar{\mathfrak{V}}_2$ on $(\mathscr{S}_1 \times Y_1) \times (\mathscr{S}_2 \times Y_2)$; algebraic "independence" of K_1 and K_2 in the sense of Lemma 2(b) is sufficient. Applying the GHC and Lemma 3 in each factor, each $\bar{\mathfrak{V}}_i$ induces a nonzero map $\Omega^{j_i}(Y_i) \to \Omega^{j_i}(\mathscr{S}_i)$; hence $\bar{\mathfrak{V}}_1 \times \bar{\mathfrak{V}}_2$ does the same $\Omega^{j_1+j_2}(Y_1 \times Y_2) \to \Omega^{j_1+j_2}(\mathscr{S}_1 \times \mathscr{S}_2)$ and so $[\mathfrak{V}_1 \times \mathfrak{V}_2]_{j_1+j_2} \neq 0$. If we make no assumption on K_1 and K_2 then there are problems:

Example 4. Let $Y_1 = Y_2 = E/\bar{\mathbb{Q}}$ be an elliptic curve and $\mathscr{V}_1 = \mathscr{V}_2 = p - o$ where $p \in E(\mathbb{C})$ is very general and o is a $\bar{\mathbb{Q}}$-point. Then $[\mathfrak{V}_1]_1 \neq 0$, $[\mathfrak{V}_2]_1 \neq 0$ but $\mathscr{V}_1 \times \mathscr{V}_2 = (p,p) - (o,p) - (p,o) + (o,o)$ is the diagonal cycle; this is $\overset{\mathrm{rat}}{\equiv} 0$ (mod 2-torsion if we did not work $\otimes\,\mathbb{Q}$).

Remark 6. Should one want to generalize to Y_1, Y_2 not defined $/\bar{\mathbb{Q}}$, the above assumptions—nontriviality of $cl^{j_i}_{Y_i}(\mathscr{V}_i)$ $(i = 1,2)$ and "independence" of the fields of definition of $\mathscr{V}_1$ and $\mathscr{V}_2$ (namely, K_1 and K_2) over $\bar{\mathbb{Q}}$—are insufficient to guarantee $cl^{j_1+j_2}_{Y_1\times Y_2}(\mathscr{V}_1 \times \mathscr{V}_2) \neq 0$. For a counterexample (in case $j_1 = j_2 = 1$) one can simply take λ_1 and λ_2 both general (and algebraically independent over $\bar{\mathbb{Q}}$) in Example 3. However, if one takes K_1 and K_2 independent over the common field of defintion of Y_1 and Y_2 (say, L), then an analogous result obviously holds for the higher cl-type invariants arising from the partial L-spreads of $\mathscr{V}_1$, $\mathscr{V}_2$, and $\mathscr{V}_1 \times \mathscr{V}_2$.

Now suppose (with Y_i def'd. $/\bar{\mathbb{Q}}$) $\langle \mathscr{V}_i \rangle \in \mathscr{L}^{j_i} CH_0\big((Y_i)_{K_i}\big)$ but $[\mathfrak{V}_i]_{j_i} = 0$, $[AJ(\mathfrak{V}_i)]_{j_i-1} \neq 0$ $(i = 1,2)$; i.e. each cycle has nontrivial higher AJ-class. The situation looks more grim here for nontriviality of the exterior product, even if we assume K_1 and K_2 independent. It will never be the case that $[AJ(\mathfrak{V}_1 \times \mathfrak{V}_2)]_{j_1+j_2-2} \neq 0$, because $\langle \mathscr{V}_1 \times \mathscr{V}_2 \rangle \in \mathscr{L}^{j_1+j_2}$ by Lemma 6. In fact, if we started with $\mathrm{trdeg}(K_i) = j_i - 1$ $(i = 1,2)$ then all the higher cycle- and AJ-classes of $\mathscr{V}_1 \times \mathscr{V}_2$ are zero; if an extension of the Bloch-Beilinson conjecture holds (see [K2] or [L1]) then this $\Rightarrow \overset{\mathrm{rat}}{\equiv} 0$.

The interesting problem is the asymmetric one: $\mathscr{V}_1$ with nontrivial higher cl, $\mathscr{V}_2$ with higher AJ-class $\neq 0$. Referring to Example 2, if we take $\mathscr{V}_1 = (p_1 - o_1) \times \cdots \times (p_m - o_m)$

(where $m < n$) and $\mathscr{V}_2 = (p_{m+1} - o_{m+1}) \times \cdots \times (p_{n-1} - o_{n-1}) \times \mathscr{W}_n$ on $Y_1 = \mathscr{C}_1 \times \cdots \times \mathscr{C}_m$ and $Y_2 = \mathscr{C}_{m+1} \times \cdots \times \mathscr{C}_n$ (resp.), then of course the product cycle is nontrivial. More generally, one expects any $\mathscr{Z}$ from Theorem 1 to work as $\mathscr{V}_2$.

Here is the strongest general result we could prove; note $\mathscr{V}$, $\mathscr{W}$ replace $\mathscr{V}_1$, $\mathscr{V}_2$. Naturally we would have preferred to assume only (say) $AJ^{j_2}_{Y_2}(\tilde{\mathscr{W}})^{\mathrm{tr}} \neq 0$; see Remark 7(iii) for a conditional improvement along these lines.

Theorem 2. *Let* Y_1, Y_2 $/\bar{\mathbb{Q}}$ *be smooth projective w./dimensions* d_1, d_2; $\langle\tilde{\mathscr{V}}\rangle \in \mathscr{L}^{j_1}CH_0\big((Y_1)_{\mathbb{C}}\big)$ *with* $cl^{j_1}_{Y_1}(\tilde{\mathscr{V}}) \neq 0$; $\langle\tilde{\mathscr{W}}\rangle \in \mathscr{L}^{j_2+1}CH_0\big((Y_2)_{\mathbb{C}}\big)$ *with* $AJ^{j_2}_{Y_2}(\tilde{\mathscr{W}})^{\mathrm{sf}} \neq 0$ *and* $cl^{j_2+1}_{Y_2}(\tilde{\mathscr{W}}) = \cdots = cl^{d_2}_{Y_2}(\tilde{\mathscr{W}}) = 0$. *Assume* $\tilde{\mathscr{V}}$, $\tilde{\mathscr{W}}$ *have resp. models over fields* $K_1, K_2 \subseteq \mathbb{C}$ *f.g.* $/\bar{\mathbb{Q}}$ *with* trdegs. t_1, t_2, *such that* $\bar{\mathbb{Q}}(K_1, K_2)$ *has* trdeg. $t_1 + t_2$. *Assume the* GHC.

Then $\langle\tilde{\mathscr{V}} \times \tilde{\mathscr{W}}\rangle \in \mathscr{L}^{j_1+j_2+1}CH_0\big((Y_1 \times Y_2)_{\mathbb{C}}\big)$ *has* $AJ^{j_1+j_2}_{Y_1\times Y_2}(\tilde{\mathscr{V}} \times \tilde{\mathscr{W}})^{(\mathrm{tr})} \neq 0$.

Proof. Let $\bar{\tilde{\mathfrak{W}}}$ be a choice of "complete" spread. Since $cl^i_{Y_2}(\tilde{\mathscr{W}}) = 0$ for $0 \leqq i \leqq d_2$, the image $[\tilde{\mathfrak{W}}]_i \in \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-d_2), \underline{H}^i(\eta_{\tilde{\mathscr{S}}_2}) \otimes H^{2d_2-i}(Y_2)\big)$ of $[\bar{\tilde{\mathfrak{W}}}]_i$ is zero for all i (automatic for $i > d_2$ since $Gr^i_{\mathscr{L}}CH^{d_2} = 0$ by [L1]). Thus $[\bar{\tilde{\mathfrak{W}}}]_i \in N^1H^i(\tilde{\mathscr{S}}_2) \otimes H^{2d_2-i}(Y_2)$ and by Deligne [D], Cor. 8.2.8, there exist irreducible codim.-1 $\bar{\mathbb{Q}}$-subvarieties S_α on $\tilde{\mathscr{S}}_2$ such that $[\bar{\tilde{\mathfrak{W}}}]_i \in \mathrm{Gy}\left\{\bigoplus_\alpha H^{i-2}(\tilde{S}_\alpha) \otimes H^{2d_2-i}(Y_2)\right\}$. Hence $[\bar{\tilde{\mathfrak{W}}}]$ is a sum of Gysin images of classes in $\mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-d_2+1), H^{2d_2-2}(\tilde{S}_\alpha \times Y_2)\big)$. By the HC, these are given by cycles; thus one may modify $\bar{\tilde{\mathfrak{W}}}$ (without affecting $\tilde{\mathfrak{W}}$) so that $[\bar{\tilde{\mathfrak{W}}}] = 0$.

By Lemma 2(b), the (complete) spread of $\tilde{\mathscr{V}} \times \tilde{\mathscr{W}}$ is just the product of spreads, $\bar{\tilde{\mathfrak{V}}} \times \bar{\tilde{\mathfrak{W}}}$. Now we specialize in both factors as in Lemma 4 and Remark 1, obtaining (with HC) $\mathscr{V}$ and $\mathscr{W}$ exactly as in the hypotheses of the proposition below. (We also have to use GHC$(1, j_1, \mathscr{S}_1)$ to get from $[\mathfrak{V}]_{j_1} \neq 0$ to the map of holomorphic forms.) According to the Proposition, $[AJ(\mathfrak{V} \times \mathfrak{W})]^{\mathrm{tr}}_{j_1+j_2} \neq 0$; and $[AJ(\tilde{\mathfrak{V}} \times \tilde{\mathfrak{W}})]^{\mathrm{tr}}_{j_1+j_2}$ maps to this under the specialization. (That this map is well-defined simply follows from well-definedness of $H^{j_1+j_2}(\tilde{\mathscr{S}}_1 \times \tilde{\mathscr{S}}_2)/N^1 \to H^{j_1+j_2}(\mathscr{S}_1 \times \mathscr{S}_2)/N^1$ [argue as in proof of Lemma 4(b)].) $\square$

So the proof has been reduced to this statement, which is what we could prove *without* assuming GHC.

Proposition. *Let* $\langle\mathscr{V}\rangle \in \mathscr{L}^{j_1}CH_0\big((Y_1)_{k_1}\big)$, $\langle\mathscr{W}\rangle \in \mathscr{L}^{j_2+1}CH_0\big((Y_2)_{k_2}\big)$ *for* $k_1, k_2 \subseteq \mathbb{C}$ *of resp.* $\mathrm{trdeg}_{/\bar{\mathbb{Q}}}$ j_1 *and* j_2. *Assume that* $\bar{\mathfrak{V}}$ *induces a nontrivial map of holomorphic* j_1*-forms* $\Omega^{j_1}(Y_1) \to \Omega^{j_1}(\mathscr{S}_1)$; *that* $[AJ(\mathfrak{W})]^{\mathrm{sf}}_{j_2} \neq 0$; *and that* $\mathscr{W}$ *has a complete spread* $\bar{\mathfrak{W}}$ *with* $[\bar{\mathfrak{W}}] = 0$. *Then* $[AJ(\mathfrak{V} \times \mathfrak{W})]^{(\mathrm{tr})}_{j_1+j_2} \neq 0$.

Proof of Proposition. Clearly the first paragraph of the proof of Theorem 1 applies (replacing $\mathscr{S}$, j by $\mathscr{S}_1$, j_1). Define $[\bar{\mathfrak{V}}]_j$, $[\bar{\mathfrak{V}}]^\vee_j$, $\mathrm{ann}([\bar{\mathfrak{V}}]_j)$ as before. Set $d = d_1 + d_2$, $j = j_1 + j_2$, $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$, $X = Y_1 \times Y_2$, $\bar{\mathfrak{Z}} = \bar{\mathfrak{V}} \times \bar{\mathfrak{W}}$, and $\partial^{-1}\bar{\mathfrak{Z}} = \bar{\mathfrak{V}} \times \partial^{-1}\bar{\mathfrak{W}}$.

We begin with the HS (omitting the obvious dual $\mathscr{K}$'s)

$$\mathscr{H}_0 = H^j(\mathscr{S}) \otimes H^{2d-j-1}(X) \supseteq H^{j_1}(\mathscr{S}_1) \otimes H^{j_2}(\mathscr{S}_2) \otimes H^{2d_1-j_1}(Y_1) \otimes H^{2d_2-j_2-1}(Y_2) = \mathscr{H}_1$$

and

$$\mathscr{G}_0 = F_h^1 H^j(\mathscr{S}) \otimes H^{2d-j-1}(X) + H^j(\mathscr{S}) \otimes F_h^{d-j} H^{2d-j-1}(X).$$

Reasoning as in the proof of Theorem 1, under $\mathscr{H}_0 \twoheadrightarrow \mathscr{H}_1$, $\mathscr{G}_0$ projects to

$$\begin{aligned}
\mathscr{G}_0 \cap \mathscr{H}_1 &= F_h^1 \{H^{j_1}(\mathscr{S}_1) \otimes H^{j_2}(\mathscr{S}_2)\} \otimes H^{2d_1-j_1}(Y_1) \otimes H^{2d_2-j_2-1}(Y_2) \\
&\quad + H^{j_1}(\mathscr{S}_1) \otimes H^{j_2}(\mathscr{S}_2) \otimes F_h^{d-j}\{H^{2d_1-j_1}(Y_1) \otimes H^{2d_2-j_2-1}(Y_2)\} \\
&=: \mathscr{N} + \mathscr{F}.
\end{aligned}$$

Now define $\tilde{\Xi} = \left(\int_{\partial^{-1}\mathfrak{Z}} (\cdot) \right) \in \{F^{j+1}\mathscr{K}_0^{\mathbb{C}}\}^\vee$, so that $[AJ(\mathfrak{Z})]_j^{\mathrm{tr}}$ projects to $\beta_0(\Xi) \in J^d(\mathscr{H}_0/\mathscr{G}_0)$; this is what we must show nonzero. Write

$$\sigma_{23} : \mathscr{H}_1 \xrightarrow{\cong} H^{j_1}(\mathscr{S}_1) \otimes H^{2d_1-j_1}(Y_1) \otimes H^{j_2}(\mathscr{S}_2) \otimes H^{2d_2-j_2-1}(Y_2)$$

for the map exchanging the 2nd and 3rd $\otimes$-factors, and define $\mathscr{H}_{\mathfrak{B}}$ by

$$\sigma_{23}(\mathscr{H}_{\mathfrak{B}}) = \mathbb{Q}[\bar{\mathfrak{B}}]_{j_1} \otimes H^{j_2}(\mathscr{S}_2) \otimes H^{2d_2-j_2-1}(Y_2).$$

Set $\tilde{\xi} = [\mathfrak{B}]_{j_1} \otimes \left(\int_{\partial^{-1}\mathfrak{W}} (\cdot) \right) \in \{F^{j+1}\mathscr{K}_{\mathfrak{B}}^{\mathbb{C}}\}^\vee$; then $\tilde{\Xi}$ annihilates $F^{j+1}\mathscr{K}_2^{\mathbb{C}}$ and agrees with $\tilde{\xi}$ on $F^{j+1}\mathscr{K}_{\mathfrak{B}}^{\mathbb{C}}$. Next, define $\mathscr{G}_1$ by

$$\sigma_{23}(\mathscr{G}_1) = H^{j_1}(\mathscr{S}_1) \otimes H^{2d_1-j_1}(Y_1) \otimes SF_h^{(1,d_2-j_2)}\{H^{j_2}(\mathscr{S}_2) \otimes H^{2d_2-j_2-1}(Y_2)\}.$$

By assumption (nontriviality of $[AJ(\mathfrak{W})]_{j_2}^{\mathrm{sf}}$), $\pi_{\mathfrak{B}}(\xi)$ is nonzero in

$$J\left(\frac{\mathscr{H}_{\mathfrak{B}}}{\mathscr{G}_1 \cap \mathscr{H}_{\mathfrak{B}}}\right) \cong \mathbb{C}[\bar{\mathfrak{B}}]_j \otimes J^{d_2}\left(\frac{H^{j_2}(\mathscr{S}_2) \otimes H^{2d_2-j_2-1}(Y_2)}{SF_h^{(1,d_2-j_2)}\{\mathrm{num}\}}\right).$$

According to Lemma 1(ii) we are done modulo showing

$$(\mathscr{N} + \mathscr{F}) \cap \mathscr{H}_{\mathfrak{B}} \subseteqq \mathscr{G}_1.$$

Projecting along the Hodge decomposition in each factor,

$$\mathrm{p}_1' : \mathscr{H}_1^{\mathbb{C}} \twoheadrightarrow H^{j_1,0}(\mathscr{S}_1, \mathbb{C}) \otimes H^{j_2,0}(\mathscr{S}_2, \mathbb{C}) \otimes H^{d_1-j_1,d_1}(Y_1, \mathbb{C}) \otimes H^{d_2-j_2-1,d_2}(Y_2, \mathbb{C})$$

kills $\mathscr{N}$ and $\mathscr{F}$. Write $\{\alpha_i\}$ for a basis of $H^{j_1,0}(\mathscr{S}_1, \mathbb{C})$, and $\{\Gamma_\ell\}_{\ell=1}^M$ for a basis of $H^{2d_2-j_2-1}(Y_2, \mathbb{C})$ s.t. $\{\Gamma_\ell\}_{\ell=1}^r \subseteqq H^{d_2-j_2-1,d_2}(Y_2, \mathbb{C})$ and $\{\Gamma_\ell\}_{\ell=r+1}^M \subseteqq F^{d_2-j_2}H^{2d_2-j_2-1}(Y_2, \mathbb{C})$. Let

$$\underline{\mathrm{n}} + \underline{\mathrm{f}} = \sum_{\ell=1}^M \sigma_{23}^{-1}([\bar{\mathfrak{B}}]_{j_1} \otimes \gamma_\ell \otimes \Gamma_\ell)$$

be an arbitrary element of $(\mathscr{N}_{\mathbb{C}}+\mathscr{F}_{\mathbb{C}})\cap\mathscr{H}_{\mathfrak{B}}^{\mathbb{C}}$, where each $\gamma_\ell\in H^{j_2}(\mathscr{S}_2,\mathbb{C})$ has a Hodge decomposition $\sum\limits_{p+q=j_2}\gamma_\ell^{(p,q)}$. Applying p_1' gives

$$0=\sum_i\sum_{\ell=1}^r\alpha_i\otimes\gamma_\ell^{(j_2,0)}\otimes v_i\otimes\Gamma_\ell;$$

since $v_1\neq 0$ this implies

$$0=\sum_{\ell=1}^r\alpha_1\otimes\gamma_\ell^{(j_2,0)}\otimes v_1\otimes\Gamma_\ell.$$

Hence for $\ell=1,\dots,r$ we have $\gamma_\ell^{(j_2,0)}=0$, i.e. $\gamma_\ell\in\overline{F^1H^{j_2}(\mathscr{S}_2,\mathbb{C})}$. It follows that

$$\sum_{\ell=1}^M\gamma_\ell\otimes\Gamma_\ell\in\overline{F^1H^{j_2}(\mathscr{S}_2,\mathbb{C})}\otimes H^{2d_2-j_2-1}(Y_2,\mathbb{C})\tag{6}$$
$$+H^{j_2}(\mathscr{S}_2,\mathbb{C})\otimes F^{d_2-j_2}H^{2d_2-j_2-1}(Y_2,\mathbb{C}).$$

An argument symmetric in Hodge types (starting from the definition of p_1', replace all (p,q)'s by (q,p)'s) shows

$$\sum_{\ell=1}^M\gamma_\ell\otimes\Gamma_\ell\in F^1H^{j_2}(\mathscr{S}_2,\mathbb{C})\otimes H^{2d_2-j_2-1}(Y_2,\mathbb{C})\tag{7}$$
$$+H^{j_2}(\mathscr{S}_2,\mathbb{C})\otimes\overline{F^{d_2-j_2}H^{2d_2-j_2-1}(Y_2,\mathbb{C})}.$$

Hence $\sum\gamma_\ell\otimes\Gamma_\ell$ lives in the intersection

$$F^1H^{j_2}(\mathscr{S}_2,\mathbb{C})\otimes F^{d_2-j_2}H^{2d_2-j_2-1}(Y_2,\mathbb{C})+\overline{F^1H^{j_2}(\mathscr{S}_2,\mathbb{C})}\otimes\overline{F^{d_2-j_2}H^{2d_2-j_2-1}(Y_2,\mathbb{C})}$$
$$=SF^{(1,d_2-j_2)}\{H^{j_2}(\mathscr{S}_2,\mathbb{C})\otimes H^{2d_2-j_2-1}(Y_2,\mathbb{C})\}$$

of (6) and (7); and so $(\mathscr{N}+\mathscr{F})\cap\mathscr{H}_{\mathfrak{B}}$ lies in [the $\mathbb{C}$-vector space] $\sigma_{23}^{-1}(\mathbb{Q}[\overline{\mathfrak{B}}]_{j_1}\otimes SF^{(1,d_2-j_2)})$. But $(\mathscr{N}+\mathscr{F})\cap\mathscr{H}_{\mathfrak{B}}$ is a HS, hence it actually lies in the largest subHS (of, say, $\mathscr{H}_{\mathfrak{B}}$) contained in $\sigma_{23}^{-1}(\mathbb{Q}[\overline{\mathfrak{B}}]_{j_1}\otimes SF^{(1,d_2-j_2)})$, which (using purity of $\mathbb{Q}[\overline{\mathfrak{B}}]_{j_1}$) is [the $\mathbb{Q}$-vector space] $\sigma_{23}^{-1}(\mathbb{Q}[\overline{\mathfrak{B}}]_{j_1}\otimes SF_h^{(1,d_2-j_2)})$. Obviously this gives containment in $\mathscr{G}_1$ and completes the proof. □

Remark 7. (i) Theorem 1 is the case $j_2=0$.

(ii) It is easy to show (by sharpening slightly the argument in the proof of Theorem 1) that the cycles $\mathscr{Z}$ with nontrivial $[AJ(\mathfrak{Z})]_j^{\mathrm{tr}}$ produced by Theorem 1 actually have nontrivial $[AJ(\mathfrak{Z})]_j^{\mathrm{sf}}$, hence would make a suitable choice of $\mathscr{W}$ for the above proposition. (If this were not the case, one would expect a stronger proposition to be true!)

(iii) Assume the GHC. Then in the statement of the above proposition, we may relax the requirement on $\mathscr{W}$ to $[AJ(\mathfrak{W})]_{j_2}^{\mathrm{tr}}\neq 0$ provided

$$F_h^1\{H^{j_1}(\mathscr{S}_1)\otimes H^{j_2}(\mathscr{S}_2)\}=F_h^1H^{j_1}(\mathscr{S}_1)\otimes H^{j_2}(\mathscr{S}_2)+H^{j_1}(\mathscr{S}_1)\otimes F_h^1H^{j_2}(\mathscr{S}_2)$$

and

$$F^{d-j}\{H^{2d_1-j_1}(Y_1)\otimes H^{2d_2-j_2-1}(Y_2)\}$$
$$= F_h^{d_1-j_1+1}H^{2d_1-j_1}(Y_1)\otimes H^{2d_2-j_2-1}(Y_2)+H^{2d_1-j_1}(Y_1)\otimes F_h^{d_2-j_2}H^{2d_2-j_2-1}(Y_2).$$

(These are decidedly *not* satisfied if e.g. $\mathscr{S}_1=\mathscr{S}_2$.)

Proof of Remark 7(iii). In this case $\mathscr{N}+\mathscr{F}$ is annihilated by the projection from $\mathscr{H}_1$ to

$$\frac{H^{j_1}(\mathscr{S}_1)}{F_h^1}\otimes\frac{H^{j_2}(\mathscr{S}_2)}{F_h^1}\otimes\frac{H^{2d_1-j_1}(Y_1)}{F_h^{d_1-j_1+1}}\otimes\frac{H^{2d_2-j_2-1}(Y_2)}{F_h^{d_2-j_2}},$$

and it follows that $\sum\gamma_\ell\otimes\Gamma_\ell$ lies in

$$F_h^1H^{j_2}(\mathscr{S}_2)\otimes H^{2d_2-j_2-1}(Y_2)+H^{j_2}(\mathscr{S}_2)\otimes F_h^{d_2-j_2}H^{2d_2-j_2-1}(Y_2).$$

Writing $\sigma_{23}(\mathscr{G}_1')$ for $H^{j_1}(\mathscr{S}_1)\otimes H^{2d_1-j_1}(Y_1)$ tensor this, $\mathscr{G}_1'$ replaces $\mathscr{G}_1$ in the above argument and $\pi_{\mathfrak{V}}(\xi)$ need only be assumed nontrivial in

$$J^d\left(\frac{\mathscr{H}_{\mathfrak{V}}}{\mathscr{G}_1'\cap\mathscr{H}_{\mathfrak{V}}}\right)\cong\mathbb{C}[\overline{\mathfrak{V}}]_{j_1}\otimes J^{d_2}\left(\underline{H}^{j_2}(\eta_{\mathscr{S}_2})\otimes\frac{H^{2d_2-j_2-1}(Y_2)}{F_h^{d_2-j_2}\{\mathrm{num}\}}\right)$$

(assuming $\mathrm{GHC}(1,j_2,\mathscr{S}_2)$ for the $\cong$). $\square$

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